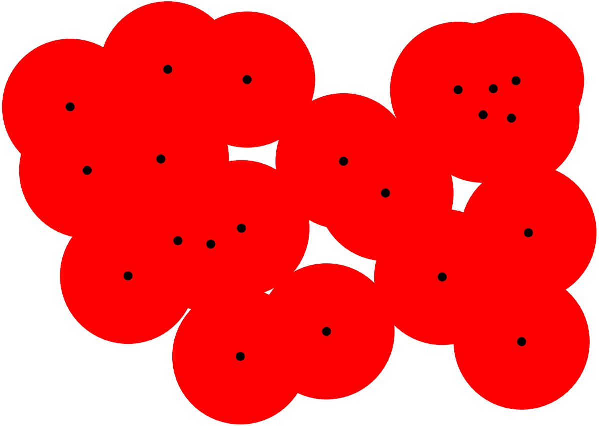
# Approximating $k$-fold filtrations using weighted Delaunay triangulations

Mickaël Buchet and Michael Kerber

TU Graz

April 4, 2019

# $k$-cover



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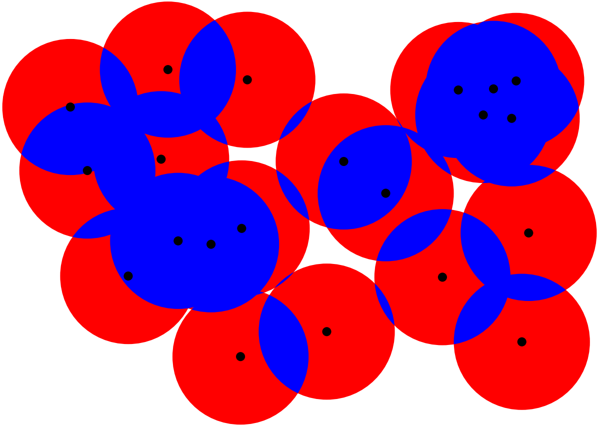

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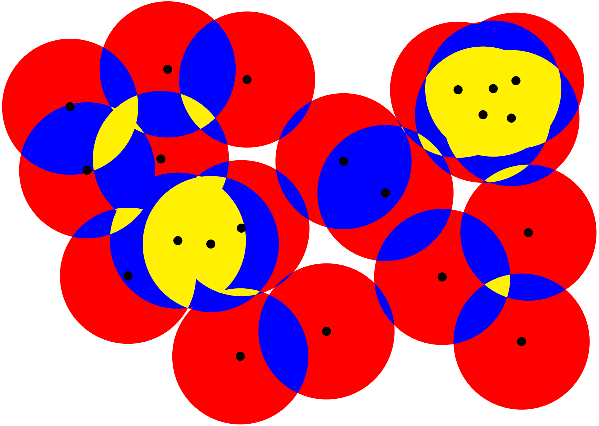

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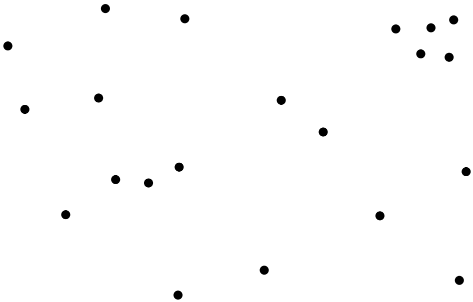
Can we approximate the result using union of balls?

# Starting easy... $k = 1$



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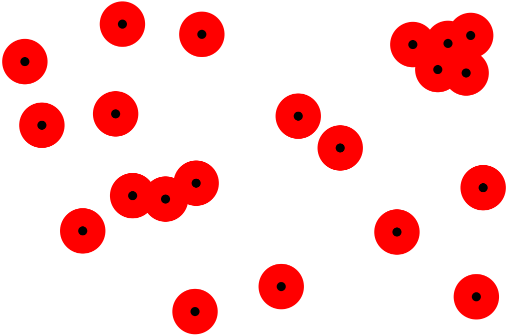

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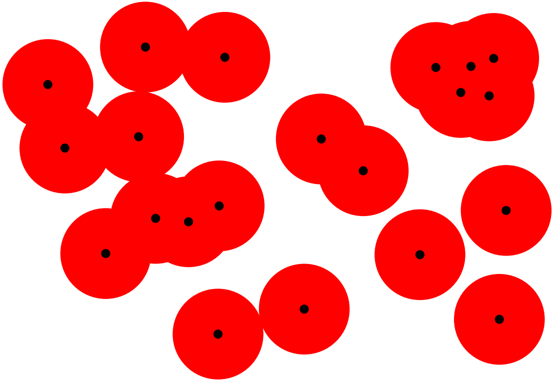

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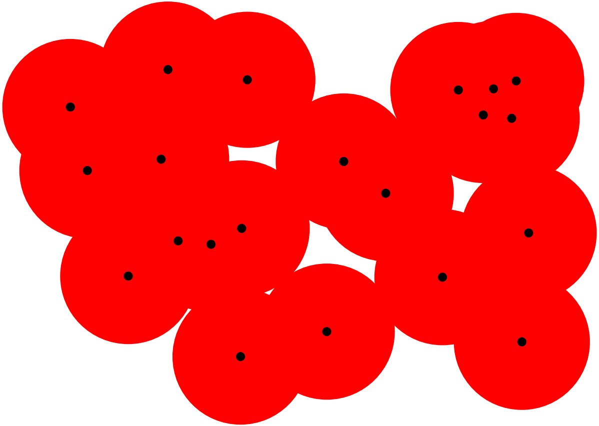

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Filtration on the Delaunay triangulation

# A functional point of view

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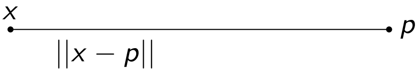

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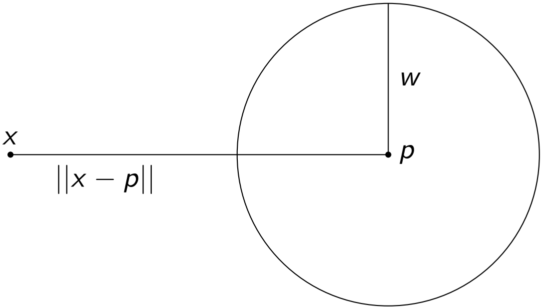

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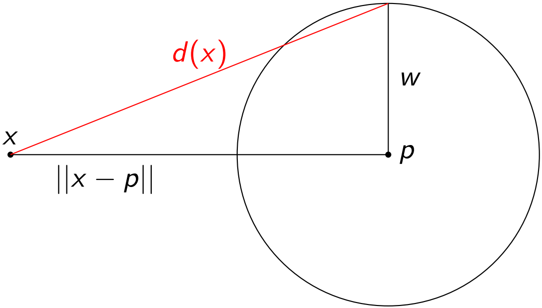
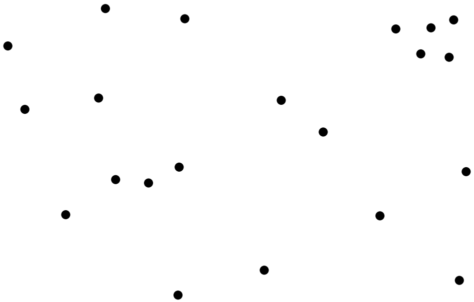


# Now for $k = 2$



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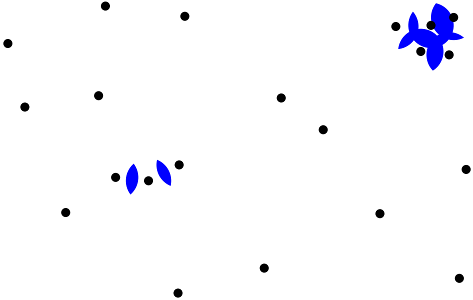

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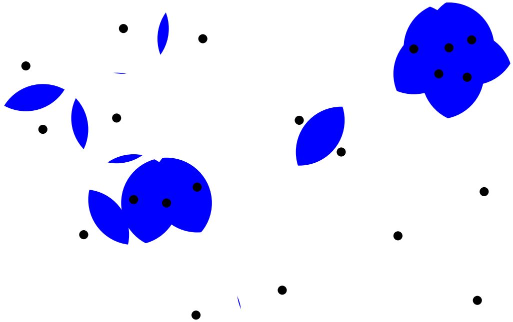

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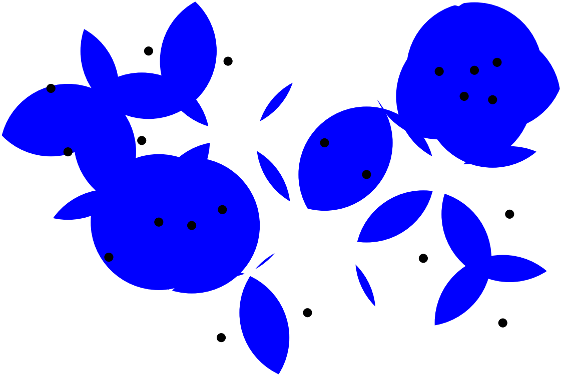
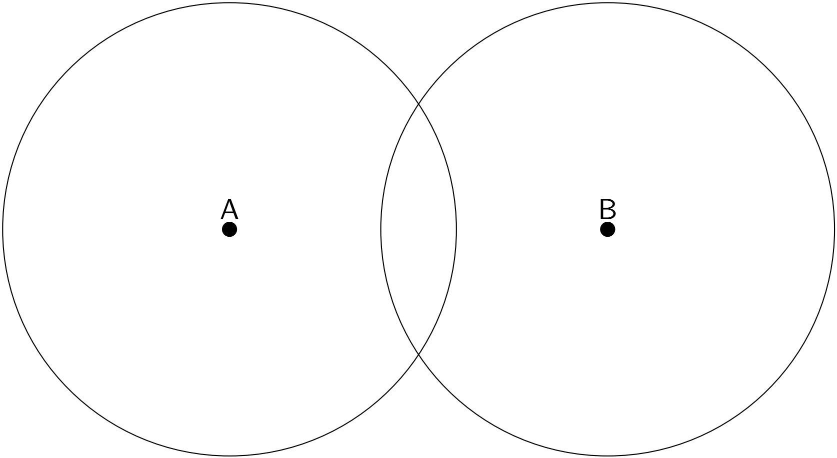


# Zooming on the lense



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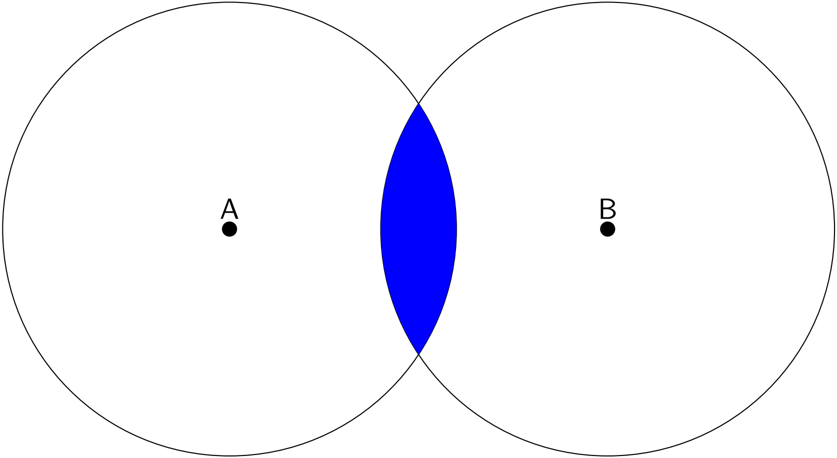

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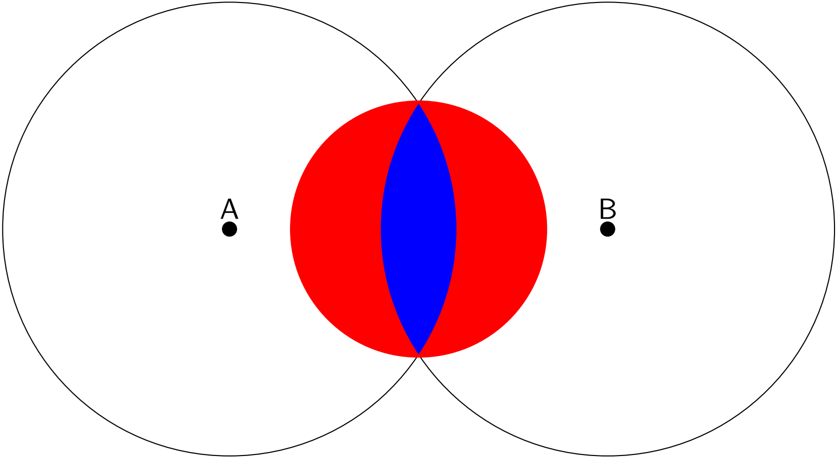

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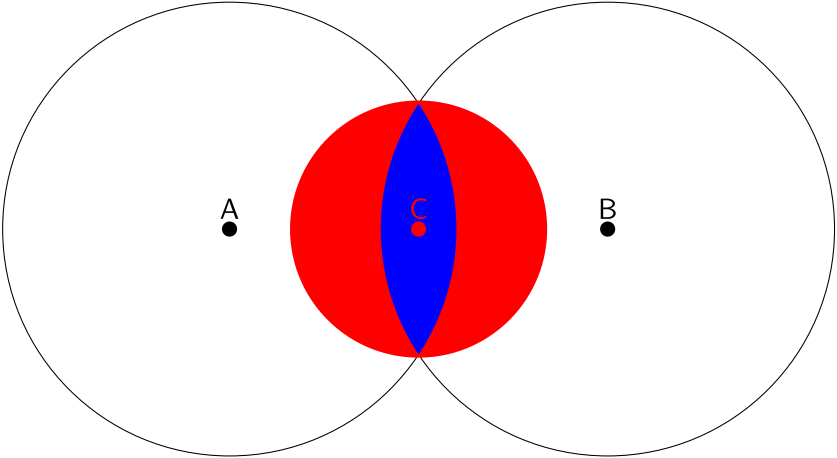

# Expressing this ball

Using previous work: distance to measure ($k$-distance).

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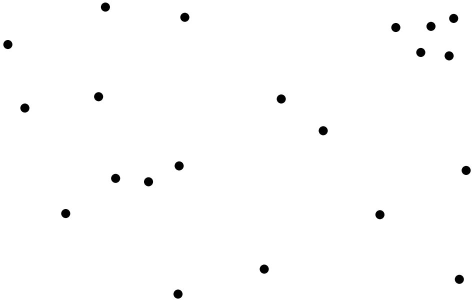
# Manicheism is too simple

# Going to $k = 3$



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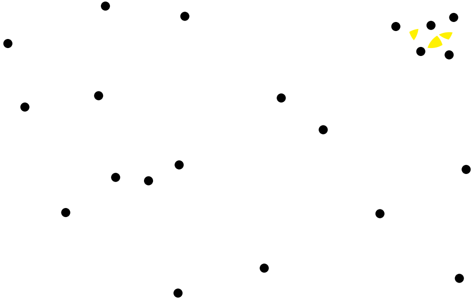

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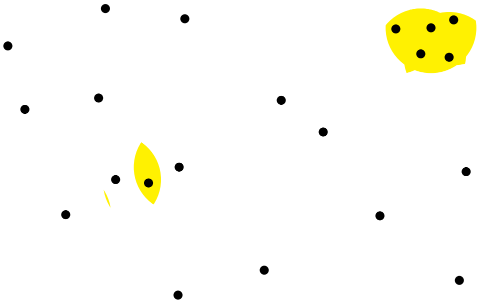

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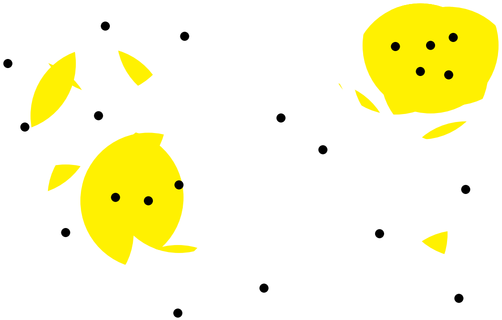

# $k$-distance and barycentres

- Considering every set of $k$-points $x_1, \ldots, x_k$ we build the barycentre $b$ and compute the weight $w^2 = \frac{1}{k} \sum ||b - x_i||^2$.

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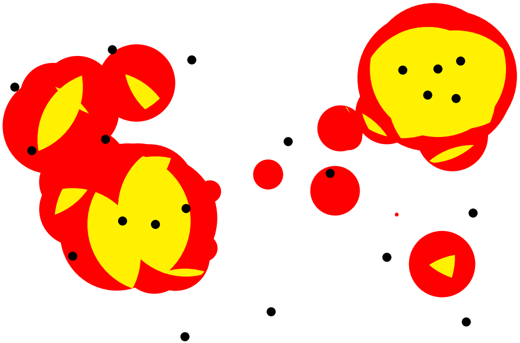
- Considering every set of $k$-points $x_1, \ldots, x_k$ we build the barycentre $b$ and compute the weight $w^2 = \frac{1}{k} \sum ||b - x_i||^2$.
- Then we look at the sublevel sets of the power distance:

$$f(x) = \min_{x_1,\ldots,x_k} \sqrt{||x - b||^2 + w^2}$$

# Barycentric over-approximation



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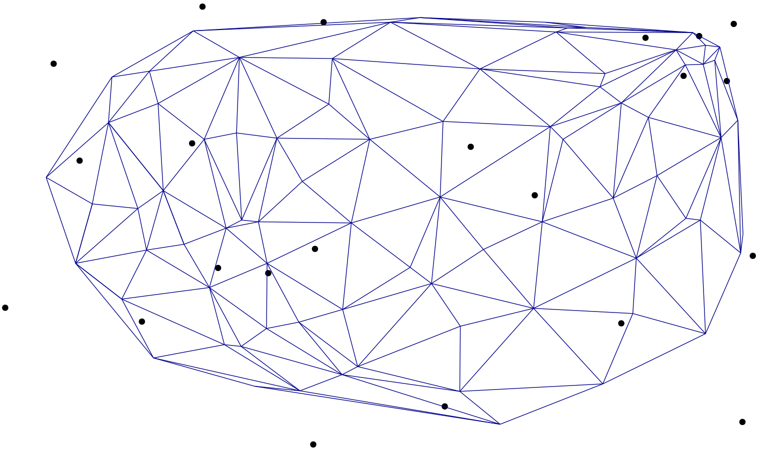
- We have a large number of balls but we can restrict to the cells of the $k$-order Voronoi diagram.
- The result is a weighted Delaunay triangulation.

# Resulting triangulation

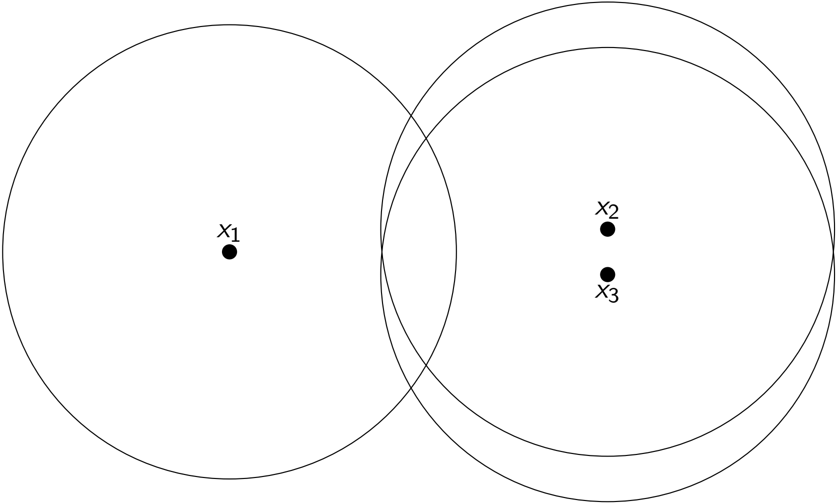


# Barycentric guarantees

**Theorem**

$$\forall k \geq 2,\ \forall \alpha > 0,\ \mathcal{K}_{\alpha}^{(k)} \subset \mathcal{A}_{\alpha} \subset \mathcal{K}_{\sqrt{k}\alpha}^{(k)}$$

# Location of barycentres



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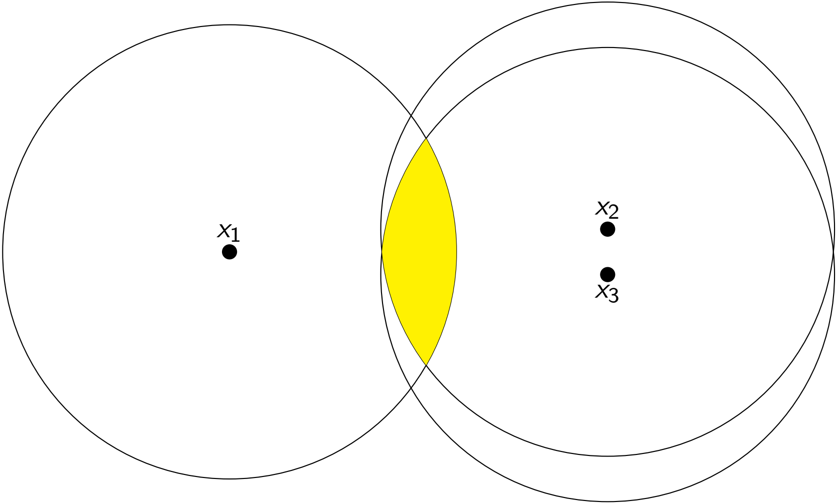

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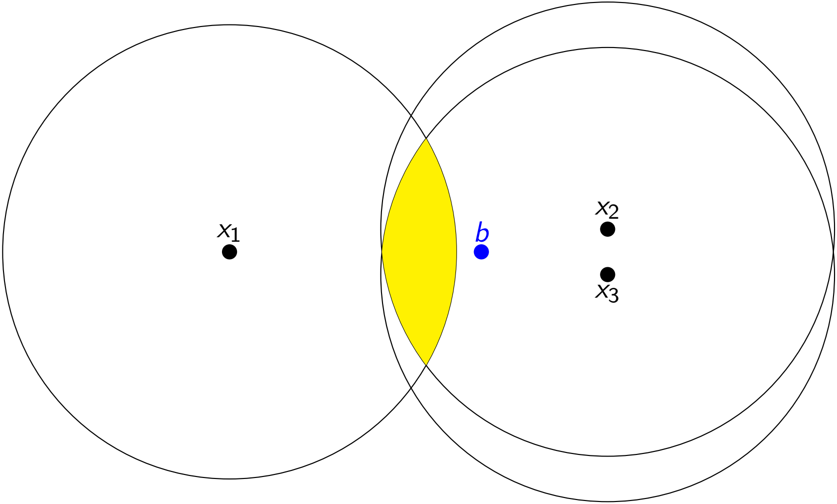

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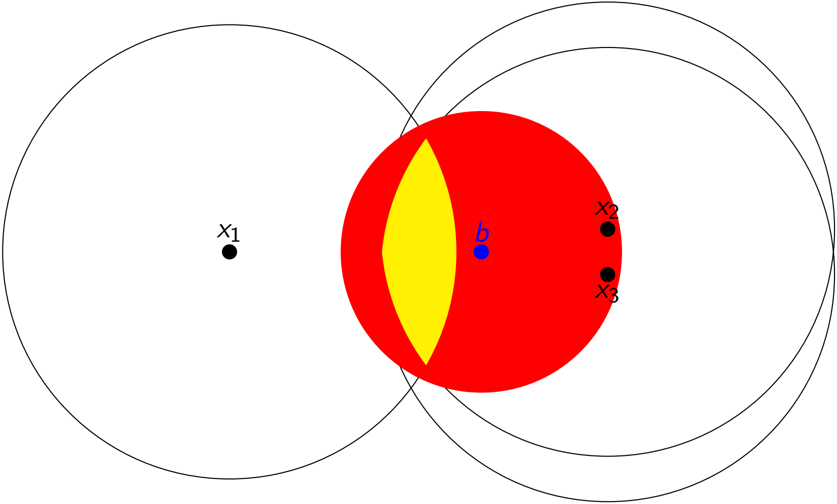

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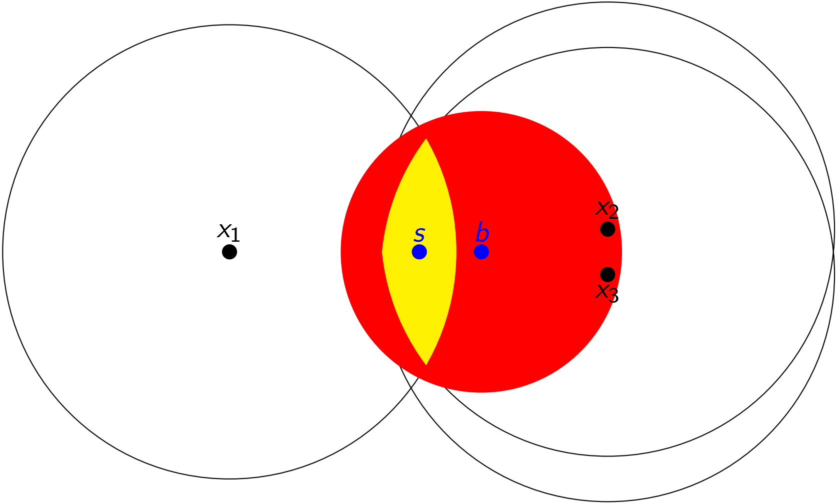

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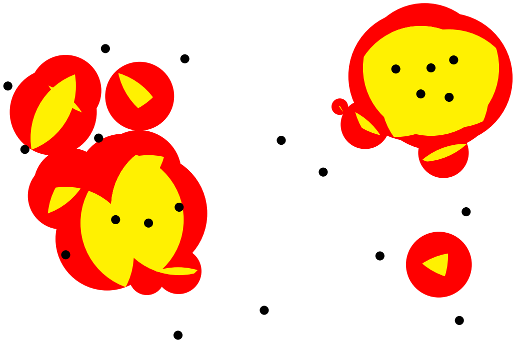
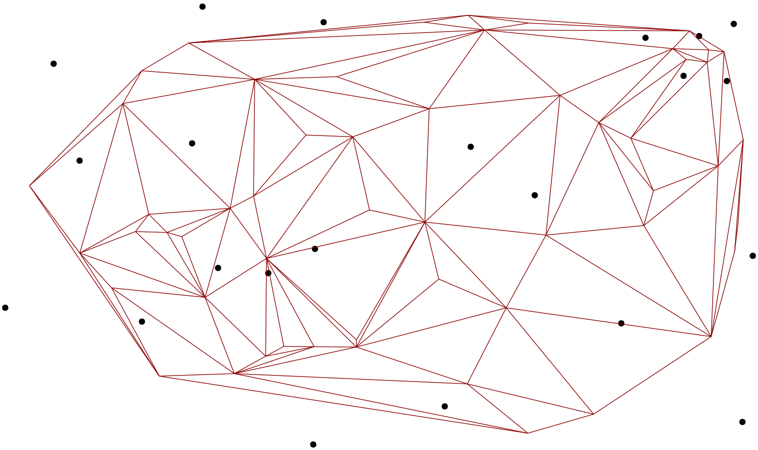
$$f(x) = \min_{T = x_1, \ldots, x_k} \sqrt{||x - s_T||^2 + R_T^2}$$

# Minimum enclosing balls covering



# MEB triangulation



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# The conjecture

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*Every non-empty cell in the MEB-diagram is also a non-empty cell in the $k$-order Voronoi diagram.*

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*Every simplex of the MEB-triangulation is also a simplex of the $k$-order Delaunay triangulation.*

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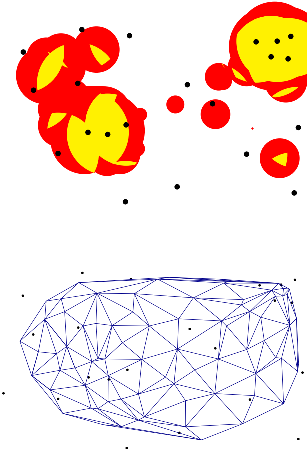
- We need to use a triangulation based on minimum enclosing balls rather than barycentres.
- This triangulation seems smaller than the $k$-order Delaunay triangulation.

# Summary (2)

Barycentres



MEBs