3D-Delaunay on points on surfaces (A Poisson sample of a smooth surface is a good sample)

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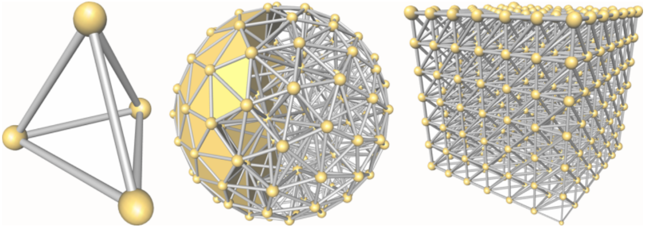
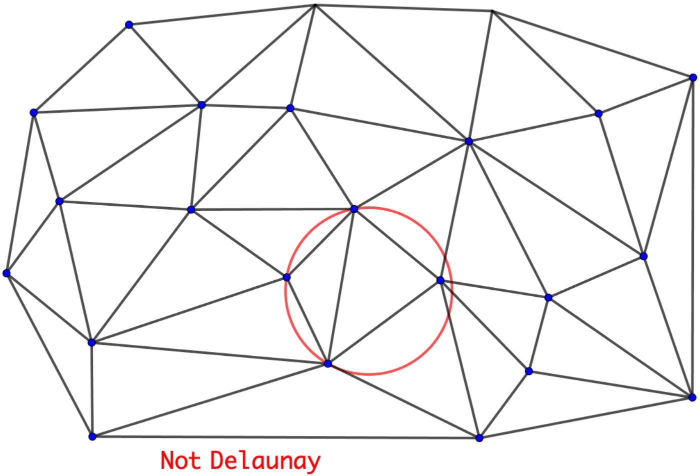
Motivation: Delaunay triangulation

Actual objective:
Computing the expected size of the Delaunay triangulation of a Poisson point process distributed on a surface.

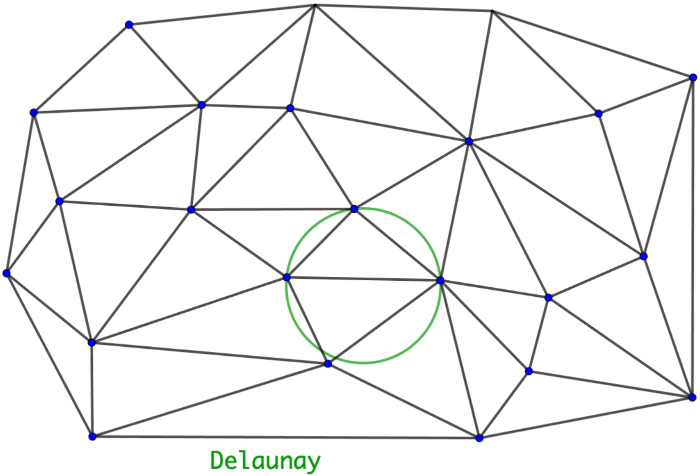


Delaunay triangulation



$\sharp\,(Del(X))$: number of edges.

Delaunay triangulation



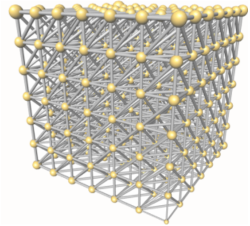
$\sharp(Del(X))$: number of edges.

State of the art

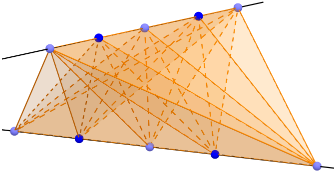
In 2D, the Delaunay triangulation has a linear number of edges like every planar graph.

In the 3D space, the size ranges:

from linear:



to quadratic:



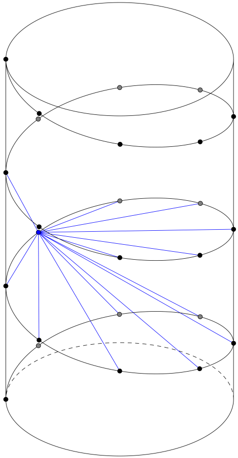
(Other example when the points are on the curve (t, t^2, t^3))

State of art, on a surface S:

Good sample $\big((\varepsilon,\kappa)$-sample$\big)$:

- at least one point in any disk of radius ε,
- at most κ points in any disk of radius ε.

Size(Del(X))	Cylinder	Generic surface
Good sample	$O(n\sqrt{n})$ [1]	$O(n \log n)$ [2]
Random Sample	$\Theta(n \log n)$ [3]	?

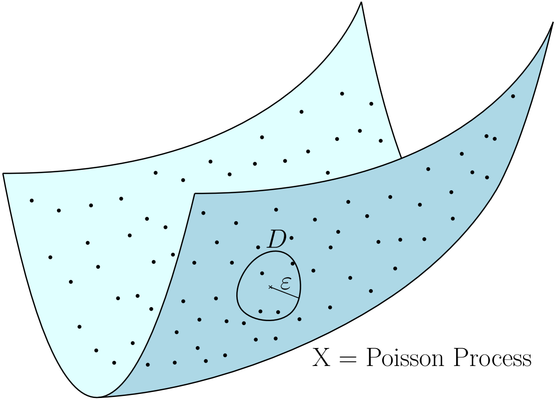


[1] Erickson

[2] Attali, Boissonnat, Lieutier

[3] Devillers, Erickson, Goaoc

On a surface, is a Poisson sample a good sample?



What is the probability that there exists:

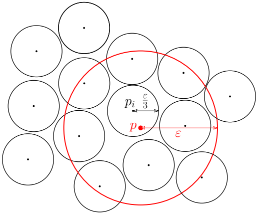
- an empty disk of radius ε? (large enough to contain points)
- a disk of radius ε that contains more than κ points?

(large enough not to be exceeded)

Sketch of the proof: $\mathbb{P}[\exists \text{ empty } \varepsilon\text{-disk}]$

α-maximal set: Maximal set of disjoint α-disks.

$\sharp(\alpha\text{-maximal set}) = O\left(\frac{1}{\alpha^2}\right)$.



$$\mathbb{P}[\exists \text{ empty } \varepsilon\text{-disk}] \leq \sum_{D_i \in \frac{\varepsilon}{3}\text{-maximal set}} \mathbb{P}[\exists \text{ empty } D_i]$$

$$= O\left(\frac{1}{\varepsilon^2}\right) e^{-\lambda Area(\frac{\varepsilon}{3}\text{-disk})}.$$

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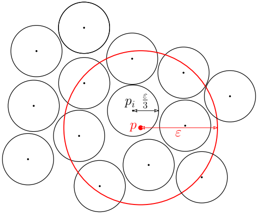

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for $\varepsilon = 3\sqrt{\frac{\log \lambda}{\lambda}}$ $\qquad = O(\lambda^{-1})$

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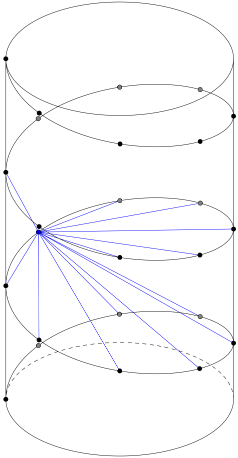
$$O\left(\left(\frac{k}{\varepsilon}\right)^2 \log(\varepsilon^{-1}) \right)$$

$$O\left(\lambda \log^2(\lambda) \right)$$



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Sketch of the proof: $\sharp\,(Del(X)) = \mathcal{O}(\lambda \log^2 \lambda)$

- Good sample: $O\left(\lambda \log^2 \lambda\right)$,
- Bad sample: $O\left(\lambda^2\right)$.

$$\begin{aligned}\mathbb{E}\left[\sharp\,(\mathrm{Del}\,(X))\right] &= \mathbb{E}\left[\sharp\,(\mathrm{Del}\,(X))\,|X \text{ good sample}\right]\mathbb{P}\left[X \text{ good sample}\right] \\ &+ \mathbb{E}\left[\sharp\,(\mathrm{Del}\,(X))\,|X \text{ not good sample}\right]\mathbb{P}\left[X \text{ not good sample}\right]\end{aligned}$$

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\mathbb{E}\left[\sharp\left(\mathrm{Del}\left(X\right)\right)\right] &= \underbrace{\mathbb{E}\left[\sharp\left(\mathrm{Del}\left(X\right)\right)|X \text{ good sample}\right]}_{= O\left(\lambda \log^2 \lambda\right)} \underbrace{\mathbb{P}\left[X \text{ good sample}\right]}_{\leq 1} \\
&+ \underbrace{\mathbb{E}\left[\sharp\left(\mathrm{Del}\left(X\right)\right)|X \text{ not good sample}\right]}_{= O\left(\lambda^2\right)} \underbrace{\mathbb{P}\left[X \text{ not good sample}\right]}_{= O\left(\lambda^{-1}\right)}
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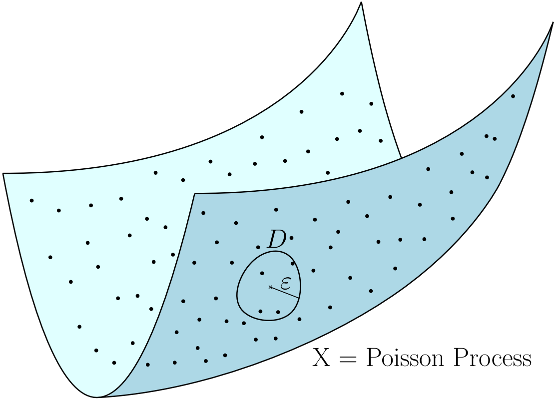
&\leq O\left(\lambda \log^2 \lambda\right) \times 1 + O(\lambda^2) \times O\left(\lambda^{-1}\right) = O(\lambda \log^2 \lambda).
\end{aligned}
$$

Current works: Computing directly with Poisson process



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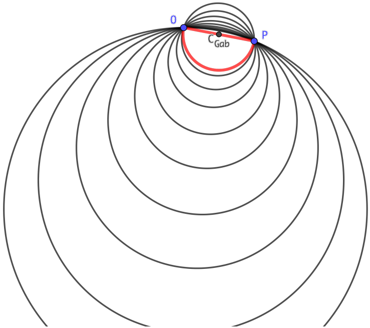
Expected size in the plane (Halfmoon graph)



Delaunay ⊂ HM-graph

$$deg(O) = \sum_{P \in X} \mathbb{1}_{[OP \in Delaunay]}$$

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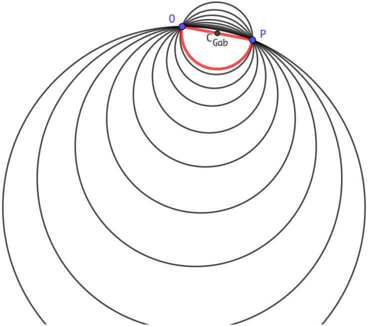

Delaunay $\subset$ HM-graph

$$deg(O) \leq \sum_{P \in X} \mathbb{1}_{[OP \in \text{HM-graph}]}$$

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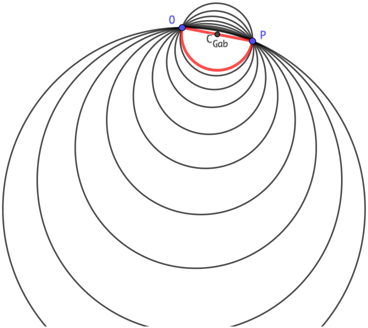

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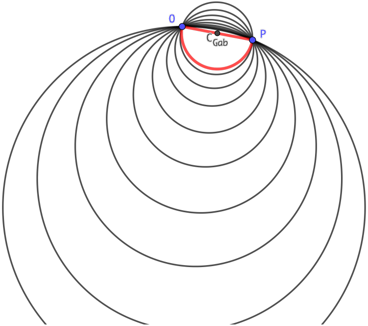

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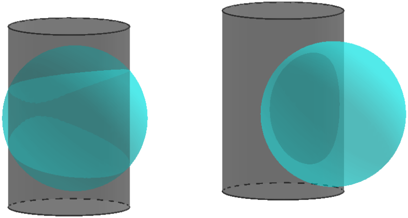
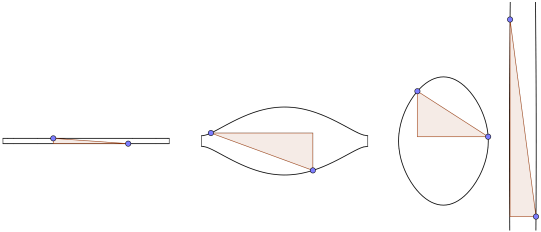
$$\mathbb{E}\left[deg(O)\right] \leq \sum_{P \in X} \mathbb{P}\left[OP \in \text{HM-graph}\right]$$

$$\leq 2 \int_{\mathbb{R}^2} exp(-\frac{\lambda}{2}(\frac{\pi(x^2+y^2)}{4}))\lambda dxdy = 16$$

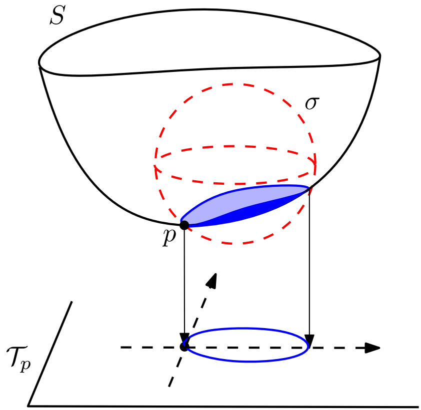
On the cylinder





$$\mathbb{E}\left[deg(O)\right] = O\left(\int_{Cyl} exp(-\lambda \frac{xy}{2}))\lambda dxdy\right) = O(\ln(\lambda))$$

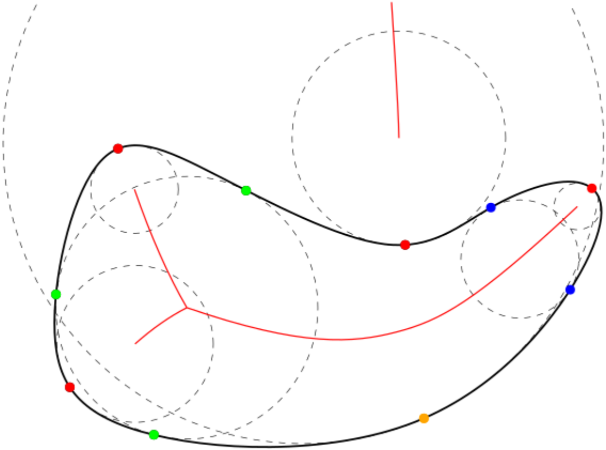
On a surface S



Empty-region graph:
Axis aligned ellipse

Expected degree:
$O(f(\text{Aspect ratio}))$

Generic surface (Almost proved)



Almost everywhere: $\mathbb{E}\left[deg\right] = O(1)$

At a red point: $\mathbb{E}\left[deg\right] = O(\ln n)$

Thanks for your attention