Numerical Algorithm for the Topology of Singular Plane Curves

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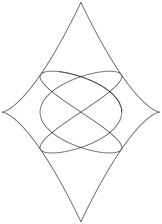
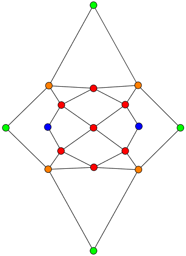
Problem

To draw plane curves with the correct topology, preserving singular points' location and distinguishing their different types.



Given curve

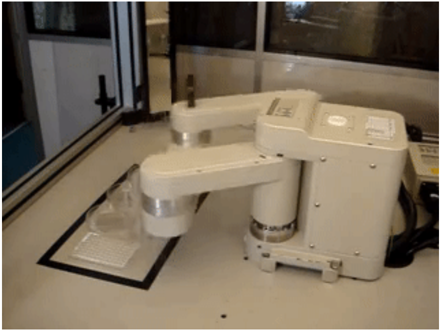


Topologically-correct graph

Example

Robot

- k_1 links
- k_2 joints
- $n = k_1 + k_2$
- 2 motors in 2 joints $\rightarrow$ 2 control variables



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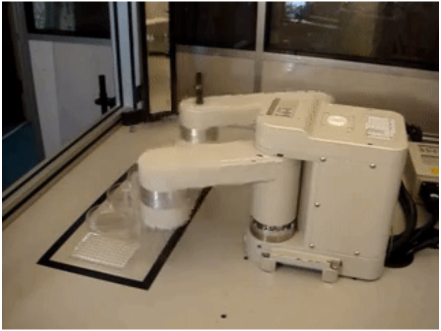

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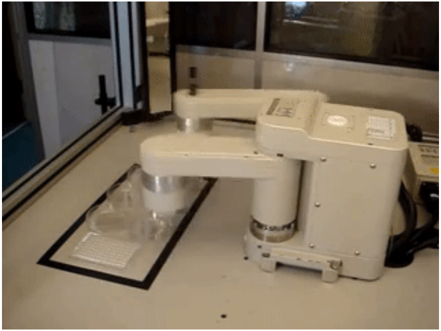

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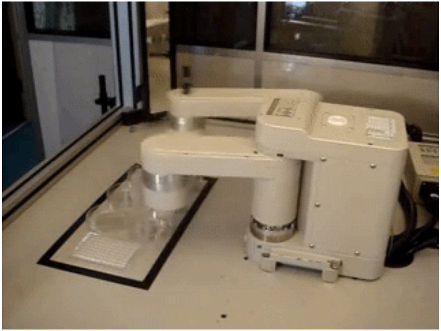

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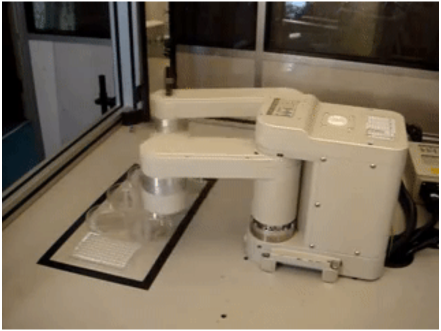

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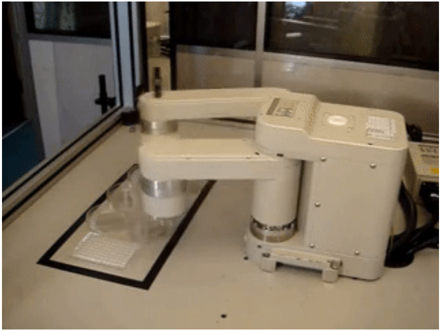

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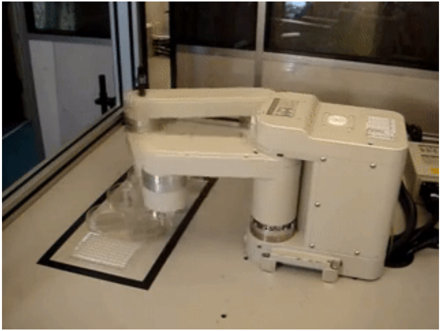

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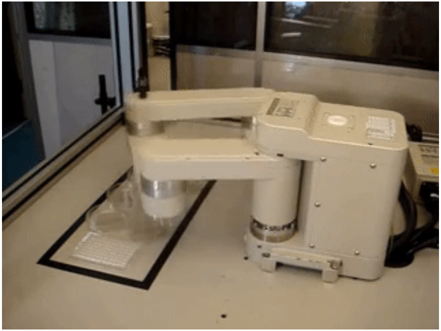

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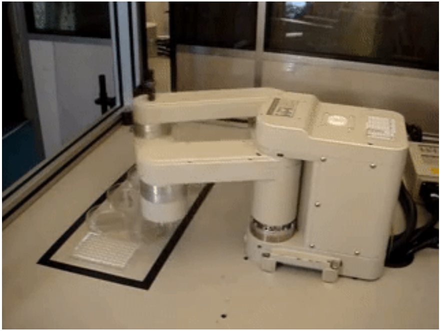

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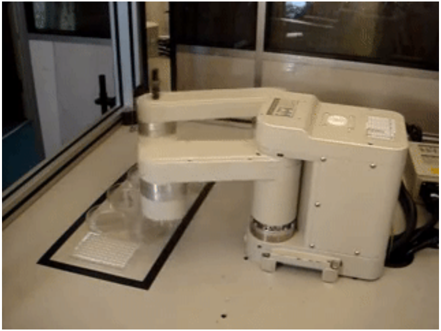

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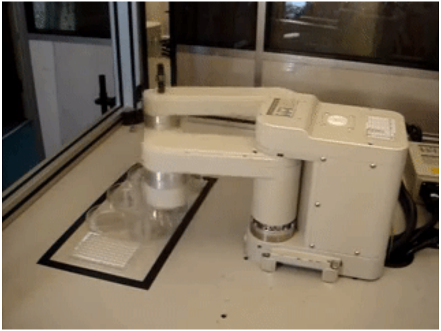

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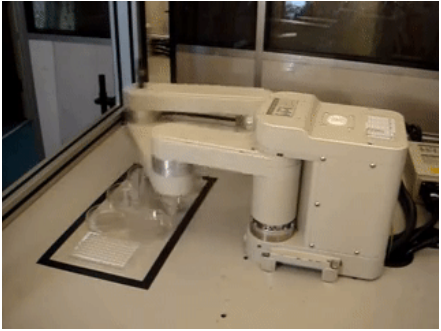

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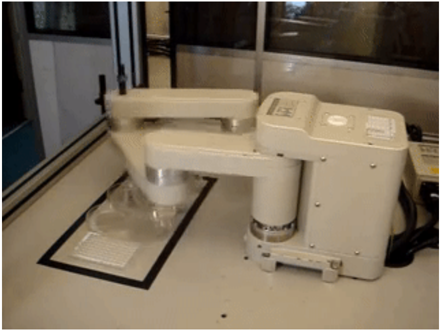

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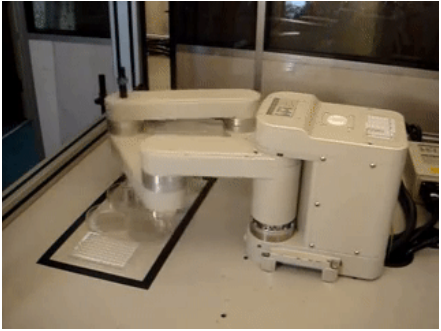

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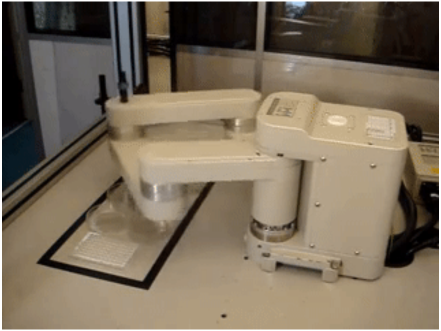

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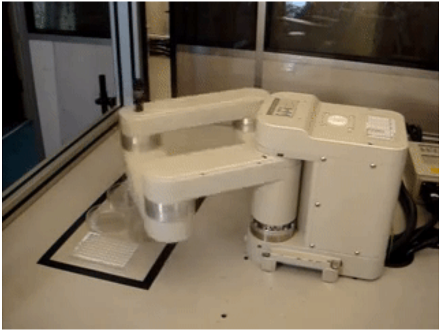

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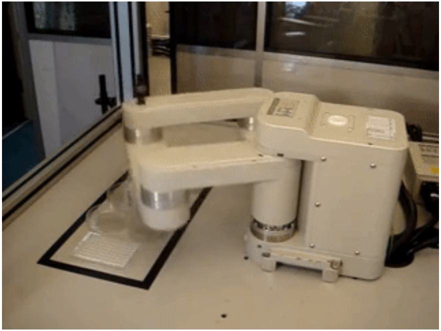

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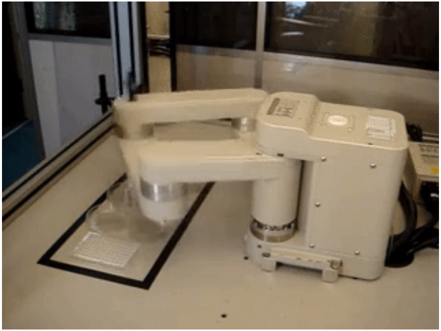

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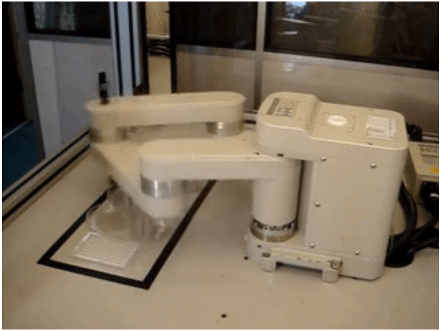

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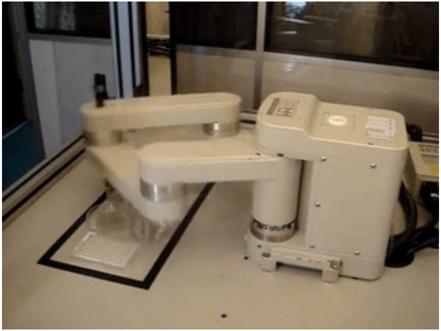

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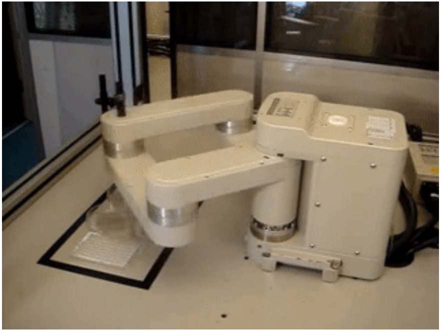

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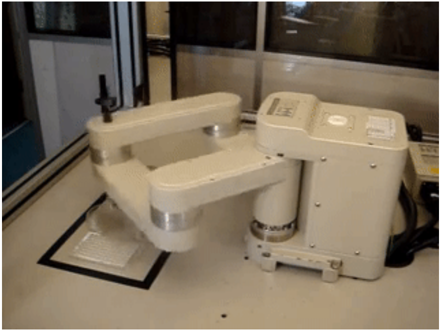

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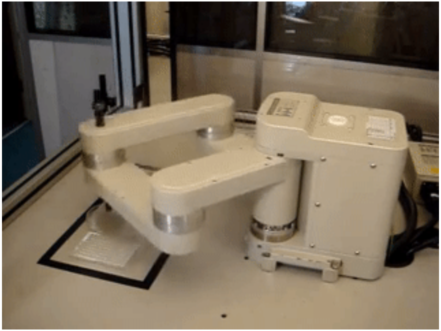

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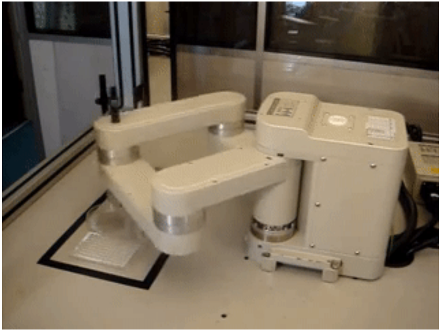

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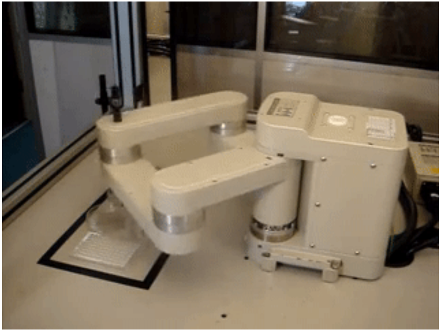

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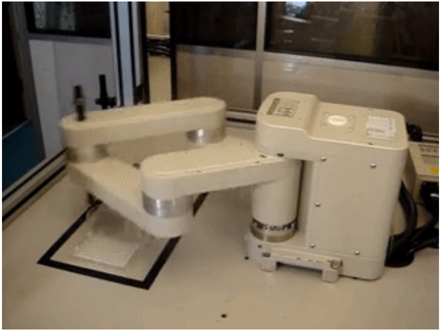

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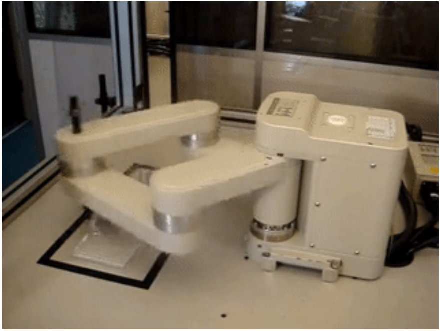

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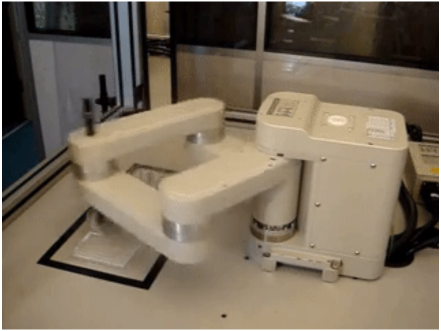

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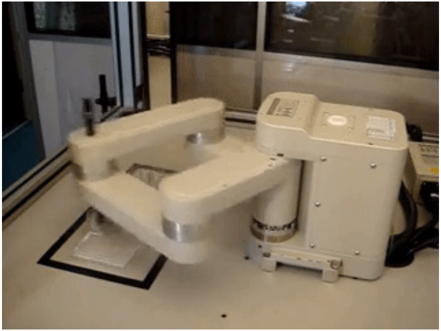

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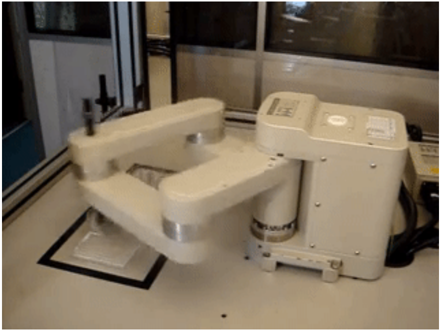

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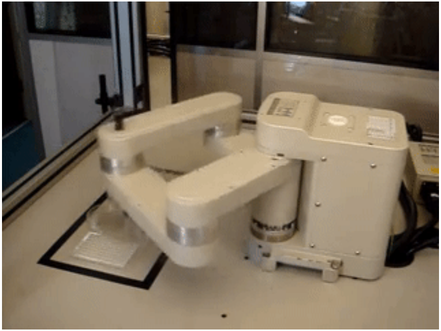

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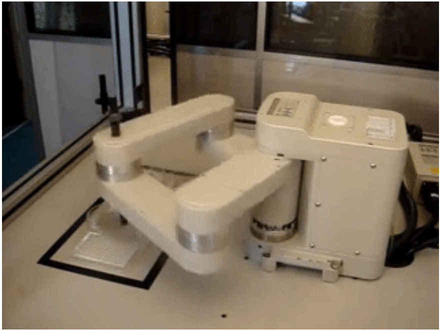

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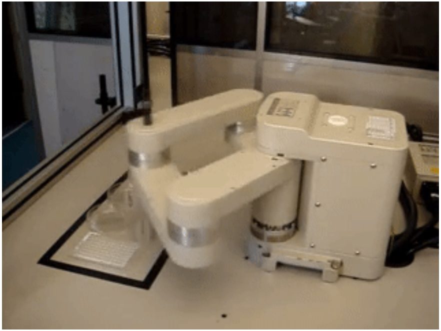

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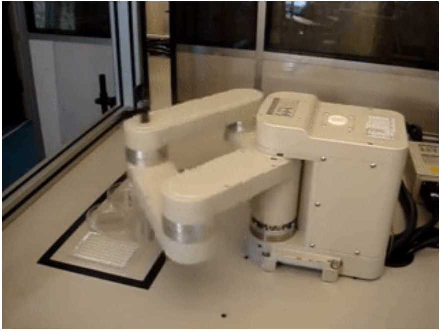

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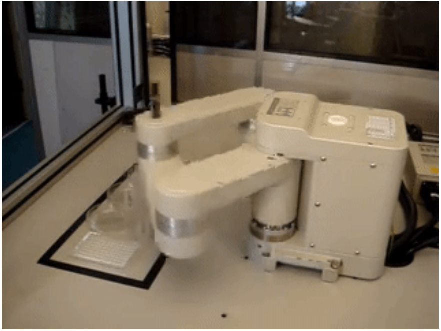

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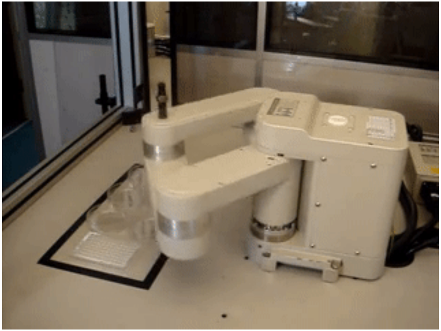

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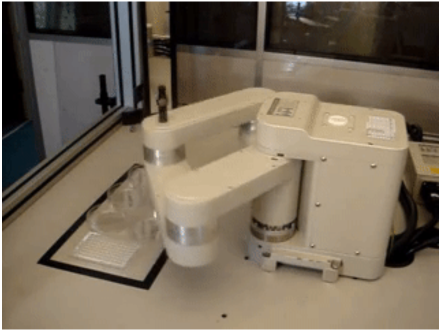

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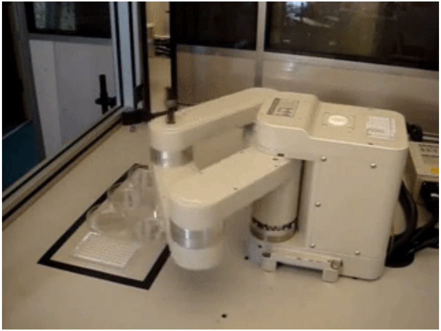

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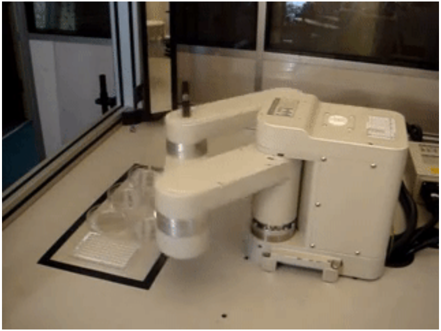

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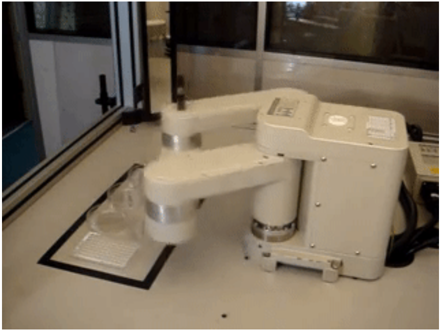

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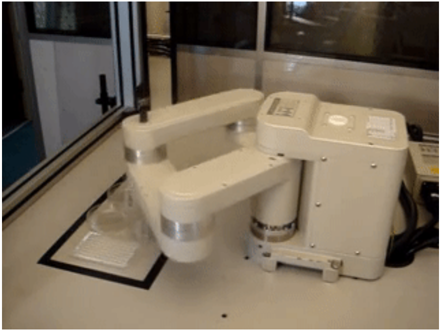

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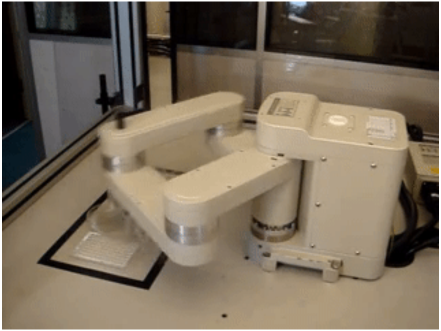

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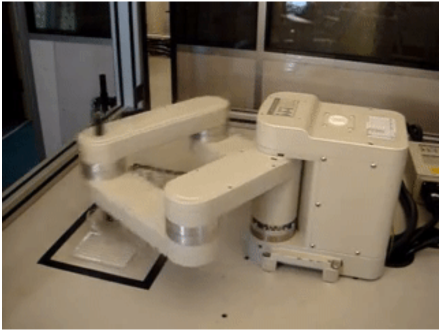

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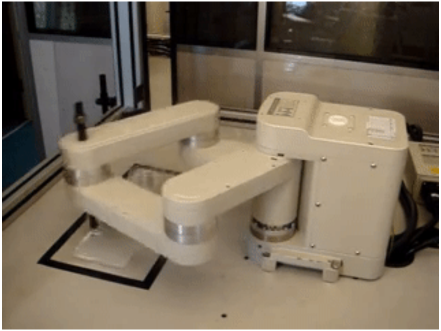

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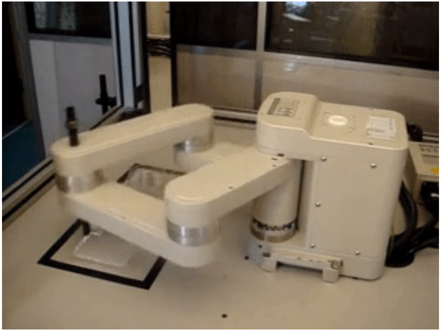

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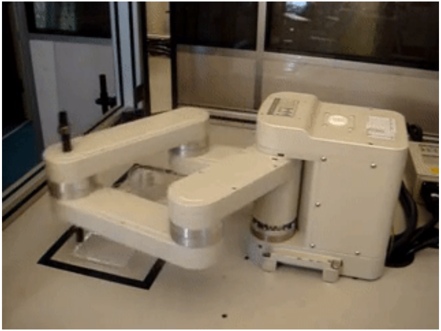

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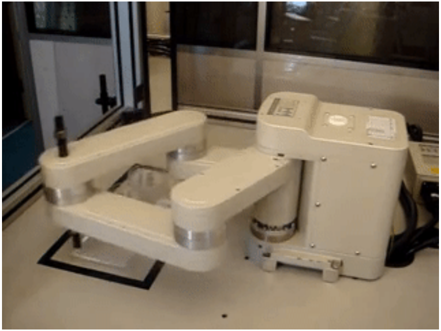

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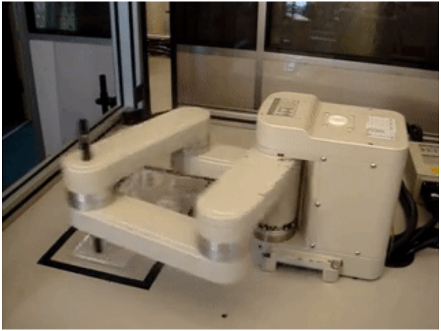

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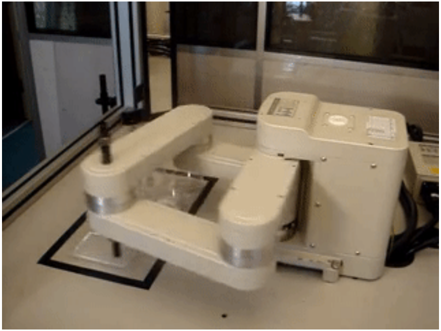
Robot

- k_1 links
- k_2 joints
- $n = k_1 + k_2$
- 2 motors in 2 joints $\rightarrow$ 2 control variables



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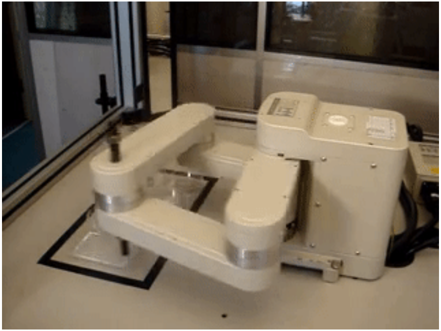

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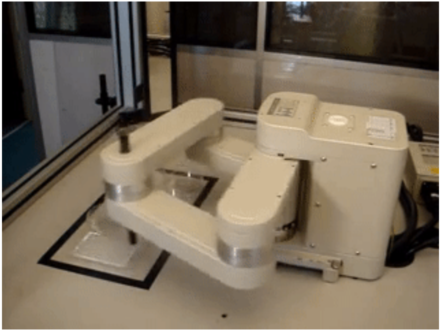

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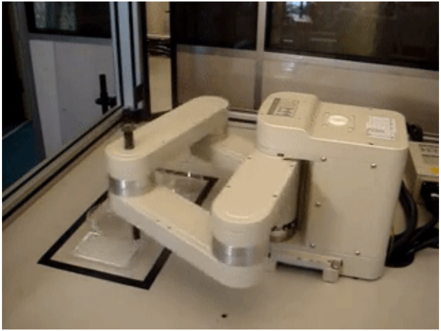

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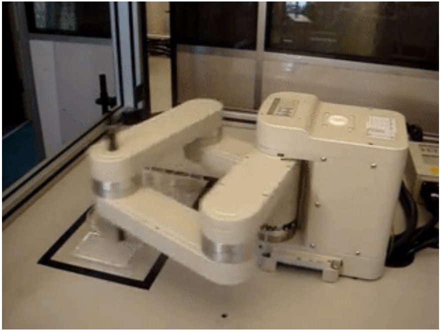

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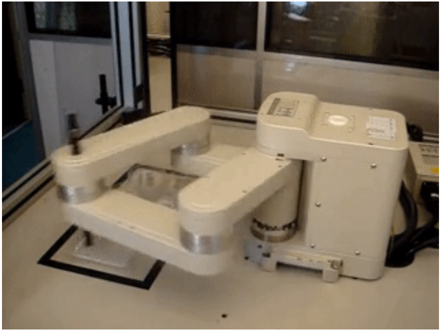

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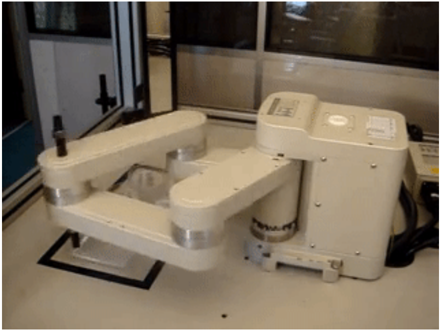

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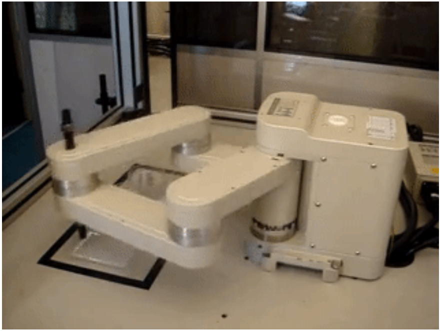

Geometric modeling of the motion: a surface in n-dim space

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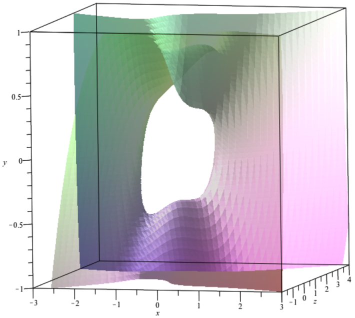
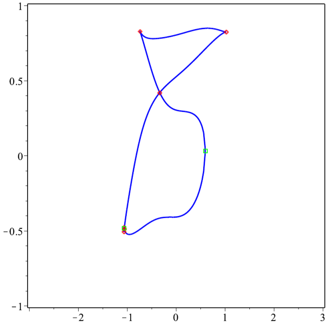
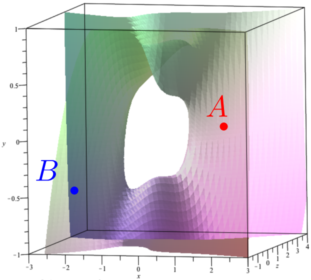


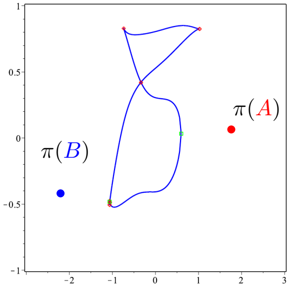


Left: a surface $f(x, y, z) = 0$. Its (smooth) silhouette curve $f = \frac{\partial f}{\partial z} = 0$.
Right: the projection of the silhouette is singular with node and cusp singularities.

Geometric modeling of the motion: surface in *n*-dim space

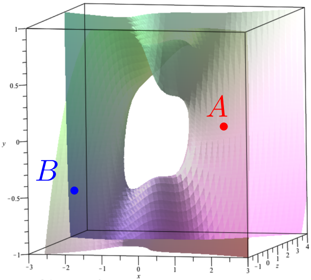


Configuration space



Control variables

Geometric modeling of the motion: surface in *n*-dim space



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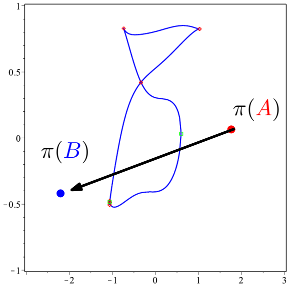

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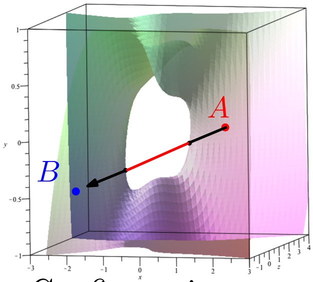

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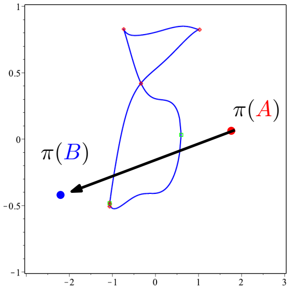

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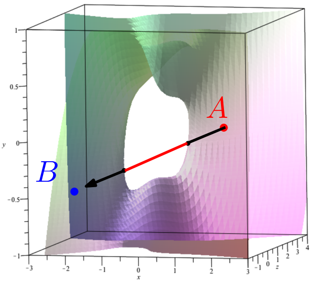

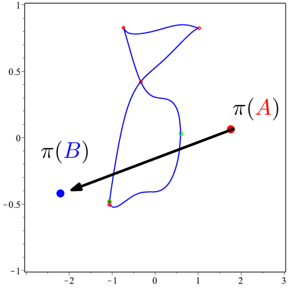

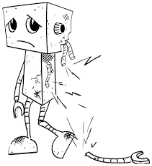

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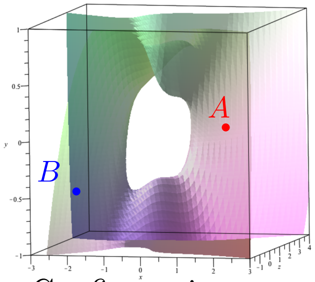

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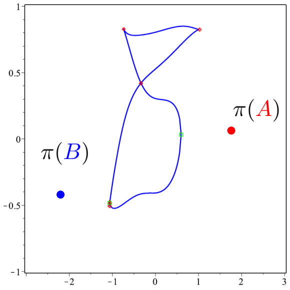

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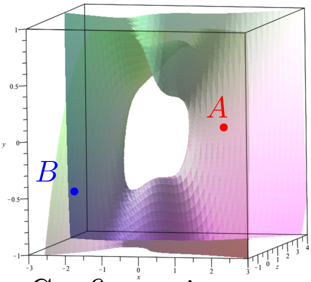

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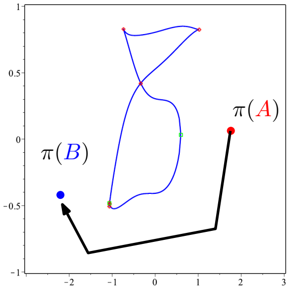

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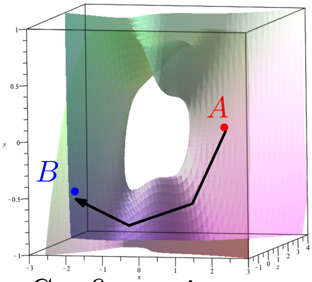

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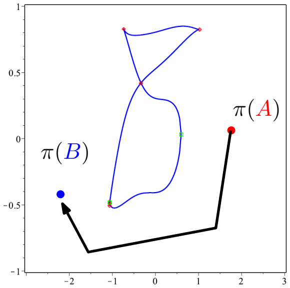

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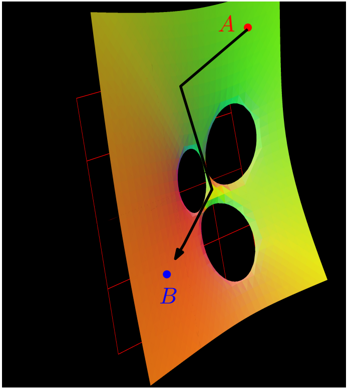
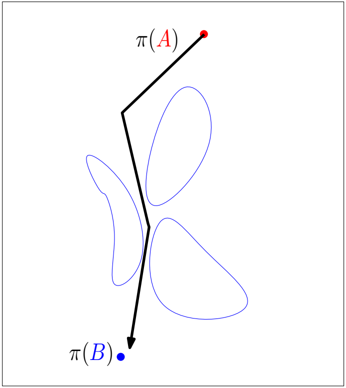
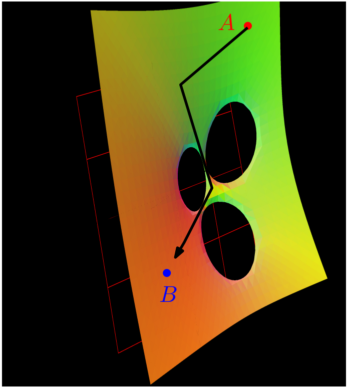
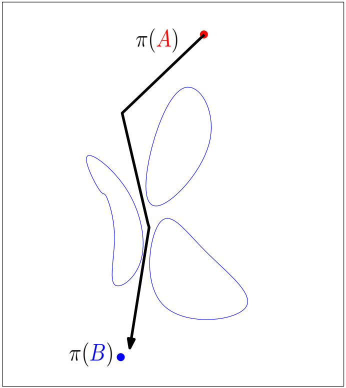

A
B


$\pi(A)$
$\pi(B)$


A
B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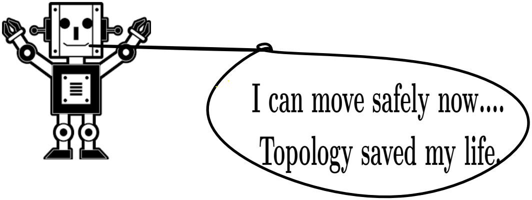

I can move safely now....
Topology saved my life.

Previous work

Question

How to compute the topology of a curve?

Previous work

Question

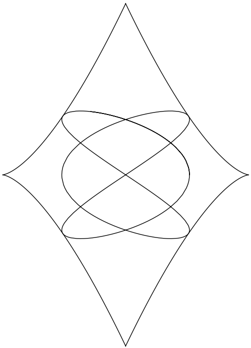
How to compute the topology of a curve?

1. Smooth curve:
   global subdivision (Snyder, 1992)
   (Plantinga and Vegter, 2004)
   (Liang et al., 2008)
   (Lin and Yap, 2011)
   continuation approaches (Beltrán and Leykin, 2013)

Question

How to compute the topology of a curve?

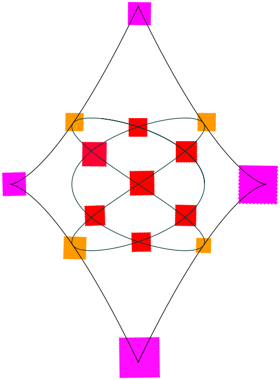
2. Singular curve:



Question

How to compute the topology of a curve?

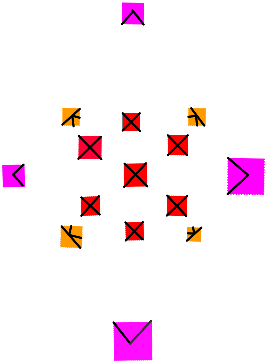
2. Singular curve:
   2.1. Isolate singular points, distinguishing their different types.



Question

How to compute the topology of a curve?

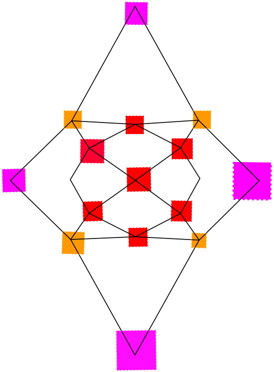
2. Singular curve:
   2.2. Compute the topology in neighborhoods of the singular points.



Question

How to compute the topology of a curve?

2. Singular curve:
    2.3. Compute the topology in the remaining smooth part.



Main challenge

It is difficult to isolate singular points.

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Available software to isolate singularities:

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Main challenge

It is difficult to isolate singular points.

Available software to isolate singularities:

- Symbolic methods: Certified... not efficient
- Numerical methods: Efficient... not certified

Our main goal is to combine the benefits of both.

How?... Certified numerical methods (Interval Newton Method)

⚠ *Condition* $\longrightarrow$ certified numerical methods

Interval Newton method: square regular system

Square: the number of variables = the number of equations.
Regular: the Jacobian matrix is full rank.

⚠ Condition → certified numerical methods

Interval Newton method: square regular system

For our problem:

We describe the singularities using a square regular system.

Square: the number of variables = the number of equations.
Regular: the Jacobian matrix is full rank.

The usual system that describes singularities

Given curve $f(x, y) = 0$

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Its singularities:

$$\left\{ \begin{array}{l} f(x,y) = 0 \\ \frac{\partial f}{\partial x}(x,y) = 0 \\ \frac{\partial f}{\partial y}(x,y) = 0 \end{array} \right\}$$

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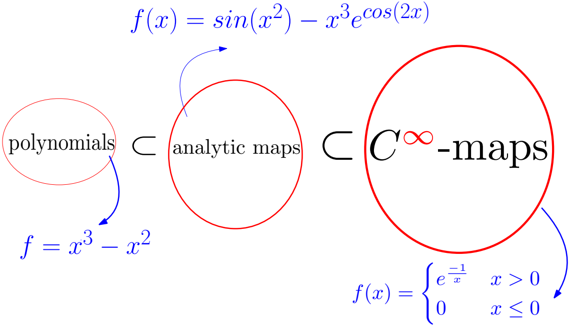
Not square!

Our approach

We restrict ourselves to the plane projection of smooth curves in higher dimensions.

Recall

C^∞-map: differentiable ∞ times



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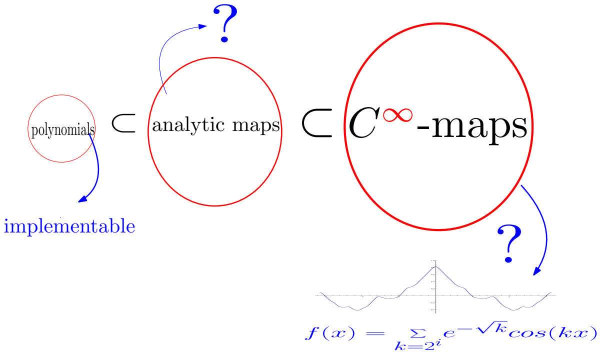

Assumptions

$\mathfrak{C}_n \subset \mathbb{R}^n$: zero set of $n-1$ C^∞-maps

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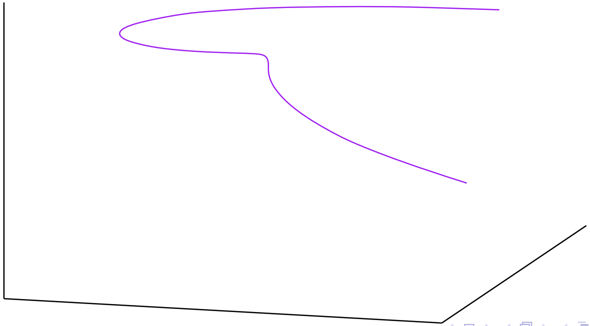
such that:

- $\mathfrak{C}_n$ smooth (actually, full-rank Jacobian).



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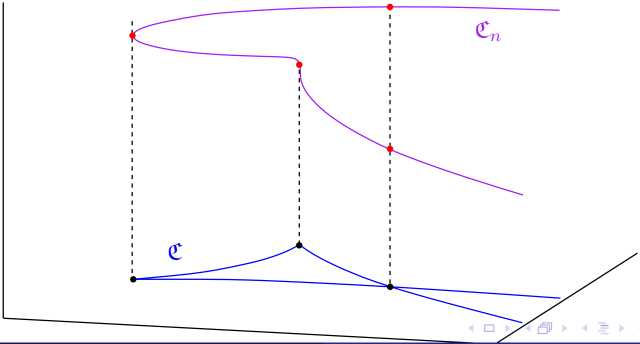
- Bad points set is finite.



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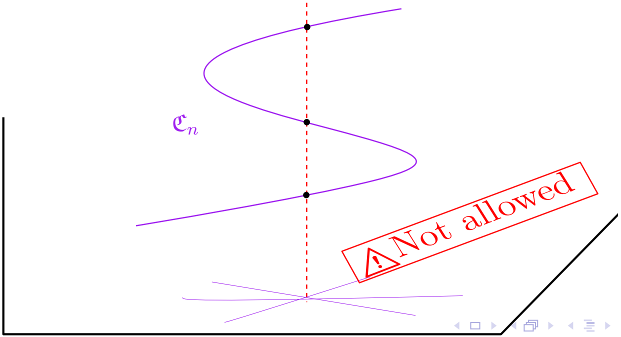
- At most two bad points have the same projection.



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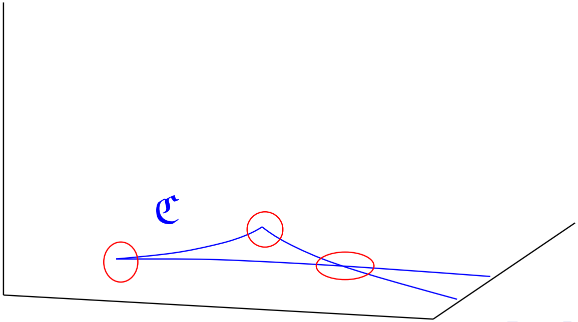
such that:

- $\mathfrak{C}_n$ smooth (actually, full-rank Jacobian).
- Bad points set is finite.
- At most two bad points have the same projection.
- Bad points project to nodes or ordinary cusps.



Results

Theorem 1

The previous assumption is generic.

Results

Theorem 2

Under the previous assumption, there exists a square regular system in $\mathbb{R}^{2n-1}$ ($\times 2$ $\# var$) that characterizes the singularities of the $2D$ curve.

Results

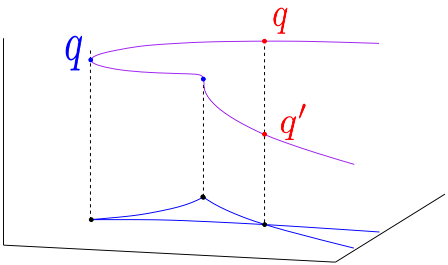
Theorem 2

Under the previous assumption, there exists a square regular system in $\mathbb{R}^{2n-1}$ ($\times 2$ #*var*) that characterizes the singularities of the 2D curve.

For $q \in \mathfrak{C}_n$:

$\pi(q)$ node $\iff$ $\pi(q) = \pi(q')$ → $\times 2$ #*var*

$\pi(q)$ cusp $\iff$ tangent of $\mathfrak{C}_n$ at q is orthogonal → $\times 2$ #*var*



Definition

Let x, r be two sets of $n-2$ real variables and t be a single real variable. For an analytic map $f : U \to \mathbb{R}$, with $U \subseteq \mathbb{R}^n$, we define the maps:,

$$S \cdot f(x_1, x_2, x, r, t) = \left\{ \begin{array}{ll} \frac{1}{2}[\, f(x_1, x_2, x + r\sqrt{t}) + f(x_1, x_2, x - r\sqrt{t})\,], & \text{for } t \neq 0 \\ f(x_1, x_2, x), & \text{for } t = 0 \end{array} \right\}$$

and

$$D \cdot f(x_1, x_2, x, r, t) = \left\{ \begin{array}{ll} \frac{1}{2\sqrt{t}}[\, f(x_1, x_2, x + r\sqrt{t}) - f(x_1, x_2, x - r\sqrt{t})\,], & \text{for } t \neq 0 \\ \nabla f \cdot (0, 0, r), & \text{for } t = 0. \end{array} \right\}$$

If $g(x) \in C^\infty +$ even $\to g(\sqrt{x}) \in C^\infty$

$\mathfrak{C}_n : P_1 = \cdots = P_{n-1} = 0$ satisfies the assumption.

Theorem 2 in more details

$(x_1, x_2) \in \mathbb{R}^2$ is a singular point in $\pi(C_P)$ if and only if there exists a solution of the following system of the form $(x_1, x_2, x, r, t) \in \mathbb{R}^{2n-1}$

$$\left\{ \begin{array}{l} S \cdot P_1(x_1, x_2, x, r, t) = \cdots = S \cdot P_{n-1}(x_1, x_2, x, r, t) = 0 \\ D \cdot P_1(x_1, x_2, x, r, t) = \cdots = D \cdot P_{n-1}(x_1, x_2, x, r, t) = 0 \\ |r|^2 = 1 \end{array} \right\}.$$

Theorem 2 in more details

Moreover, the same system is regular at its solutions.

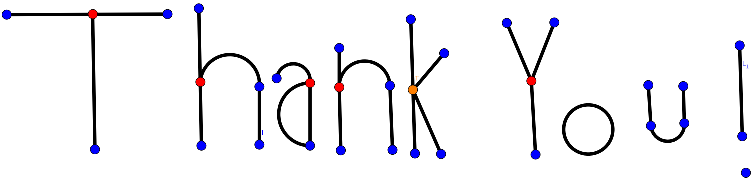
$$\left\{ \begin{array}{l} S \cdot P_1(x_1, x_2, x, r, t) = \cdots = S \cdot P_{n-1}(x_1, x_2, x, r, t) = 0 \\ D \cdot P_1(x_1, x_2, x, r, t) = \cdots = D \cdot P_{n-1}(x_1, x_2, x, r, t) = 0 \\ |r|^2 = 1 \end{array} \right\}.$$

Summary

- Efficient certified methods to isolate singularities of plane curves.
- Not only polynomials but also smooth maps.
- Assumption+No cusp $\rightarrow$ stable singularities: approximation of $\mathfrak{C}_n$ gives the same topology of $2D$ curve.

Future work

- Checking the assumption efficiently.
  - Polynomials: done
  - More general maps: ?
- Proving that a generic silhouette curve satisfies our assumption
- Implementing the algorithms
- Computing the topology of generic singular surfaces in $\mathbb{R}^n$



Beltrán, C. and Leykin, A. (2013). Robust certified numerical homotopy tracking. *Foundations of Computational Mathematics*, 13(2):253–295.

Liang, C., Mourrain, B., and Pavone, J.-P. (2008). Subdivision methods for the topology of 2d and 3d implicit curves. In *Geometric modeling and algebraic geometry*, pages 199–214. Springer, Berlin.

Lin, L. and Yap, C. (2011). Adaptive isotopic approximation of nonsingular curves: The parameterizability and nonlocal isotopy approach. *Discrete Comput. Geom.*, 45(4):760–795.

Plantinga, S. and Vegter, G. (2004). Isotopic approximation of implicit curves and surfaces. In *Proceedings of the 2004 Eurographics/ACM SIGGRAPH Symposium on Geometry Processing*, SGP '04, pages 245–254, New York, NY, USA. ACM.

Snyder, J. M. (1992). Interval analysis for computer graphics. *SIGGRAPH Comput. Graph.*, 26(2):121–130.