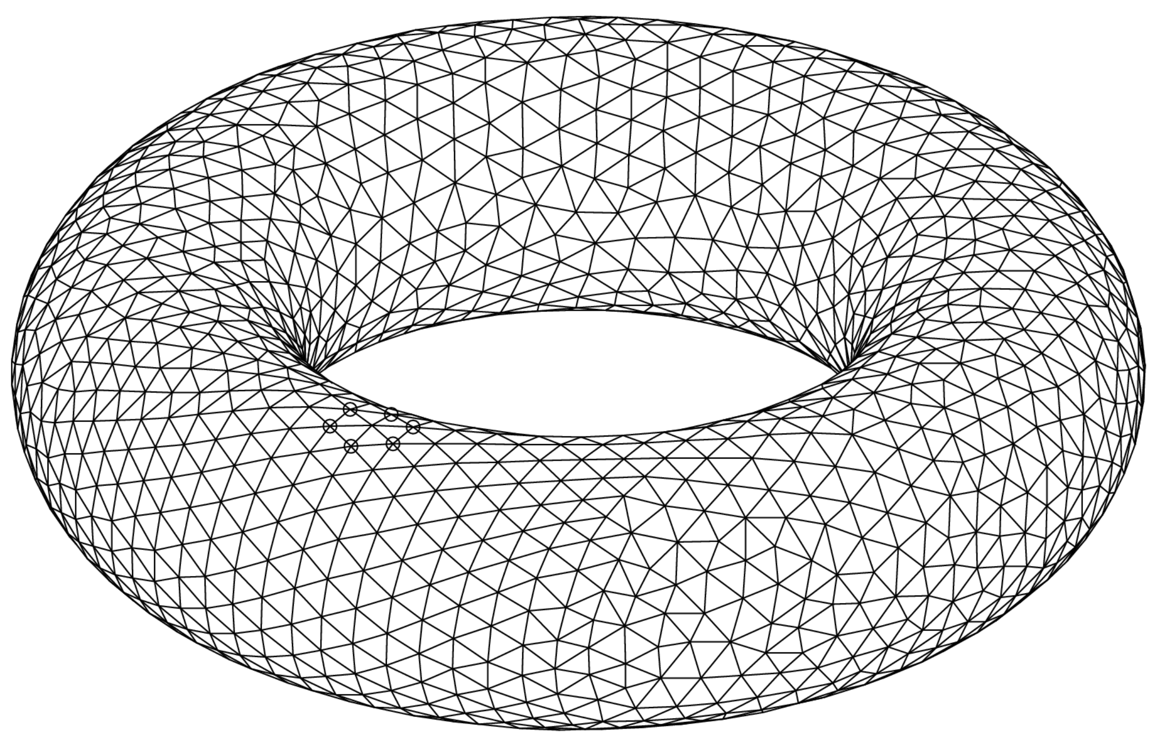
Local conditions for triangulating submanifolds of euclidean space

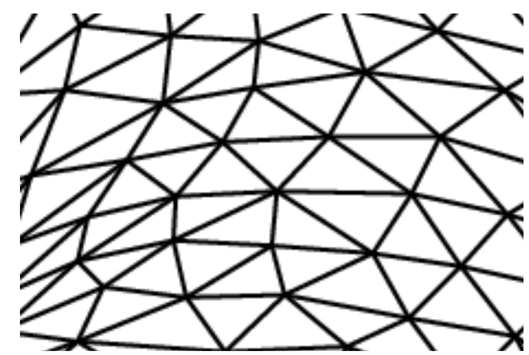
Jean-Daniel Boissonnat, Ramsay Dyer, Arijit Ghosh,
André Lieutier, and Mathijs Wintraecken



Local conditions for triangulating submanifolds of euclidean space

There are several algorithms that, given a points sample, produce a manifold triangulation.
In particular:

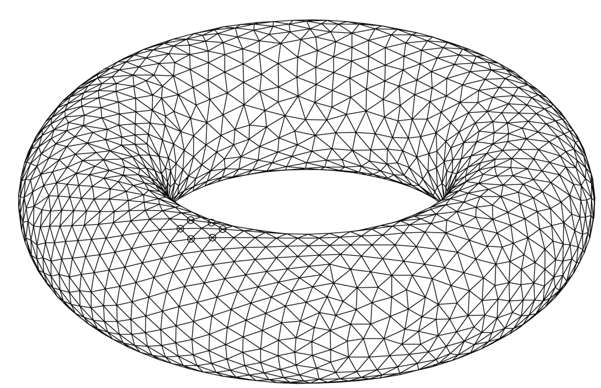
1) Tangential Delaunay Complex (Boissonnat et al.)
2) Support of « some » simplicial cycle



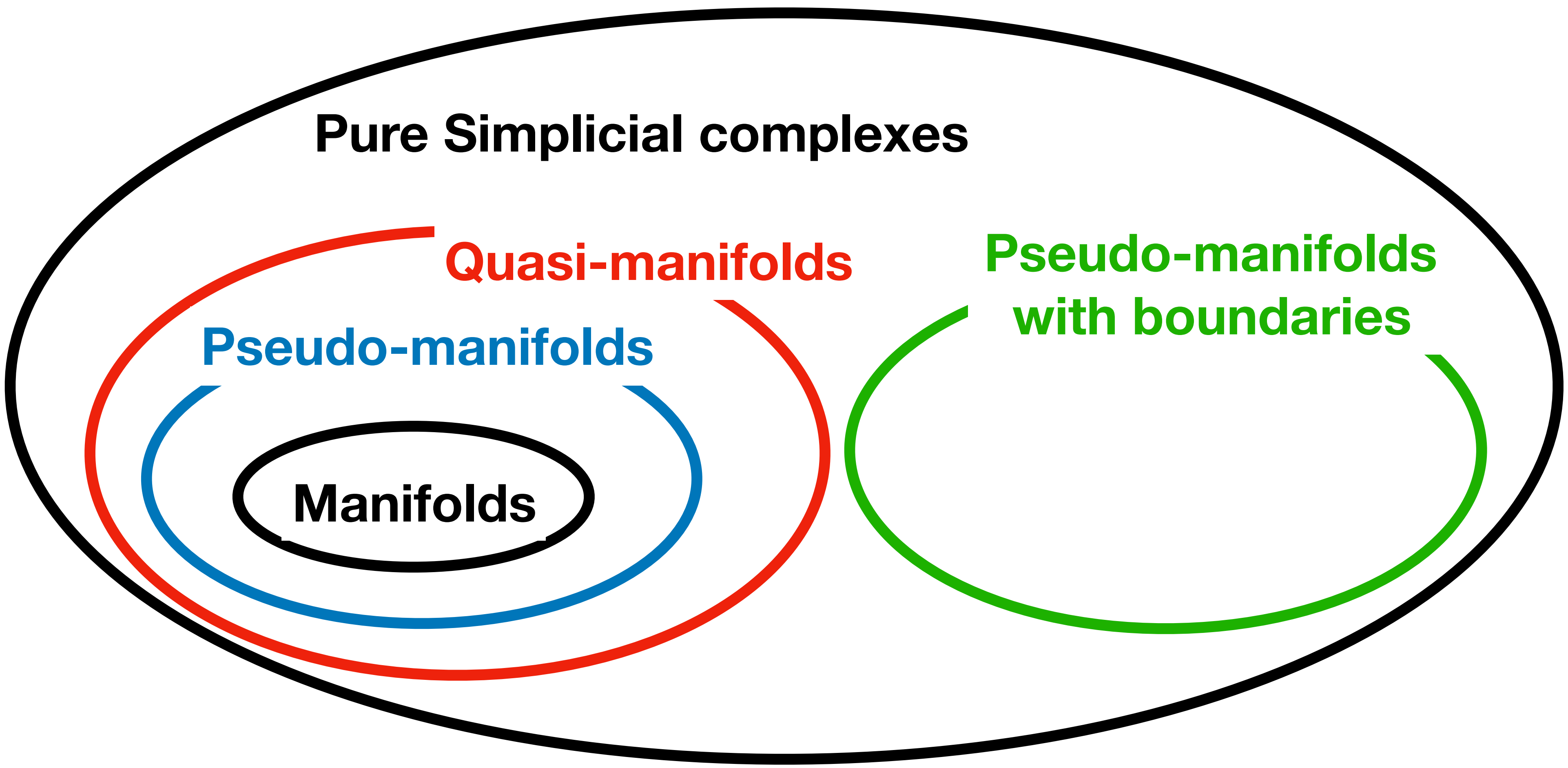
Local conditions on a simplicial complex

Today 's presentation

Global homeomorphism
(= submanifolds triangulation)



Our simplicial complex zoo



$n =$ number of full-dimensional cofaces of $(m-1)$-simplices

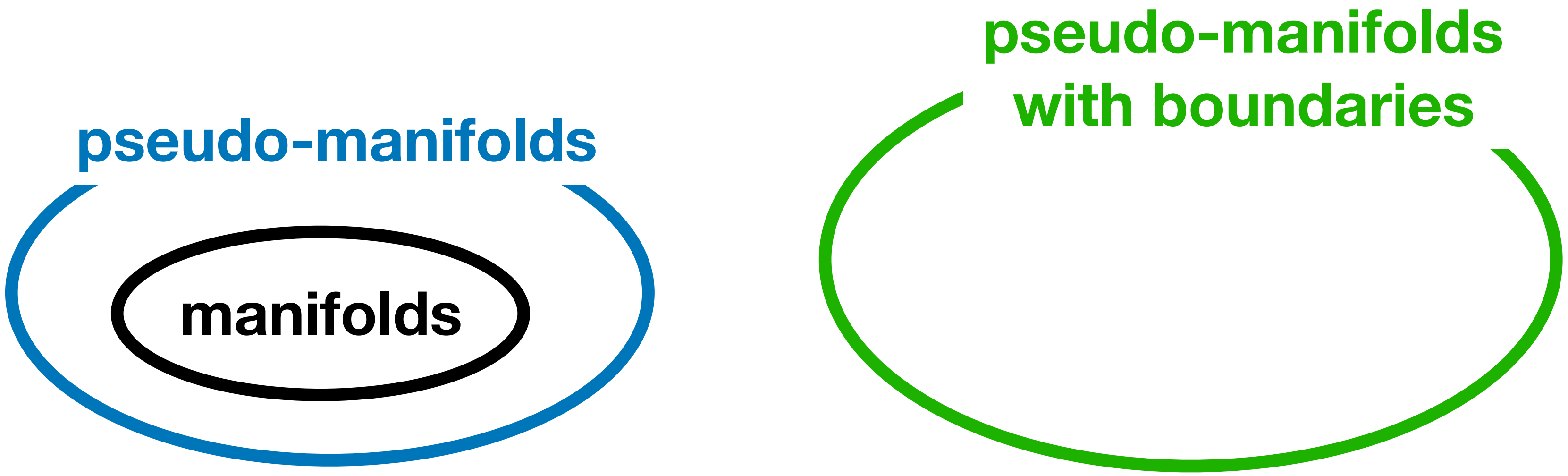
Quasi-manifold: $n \geq 2$

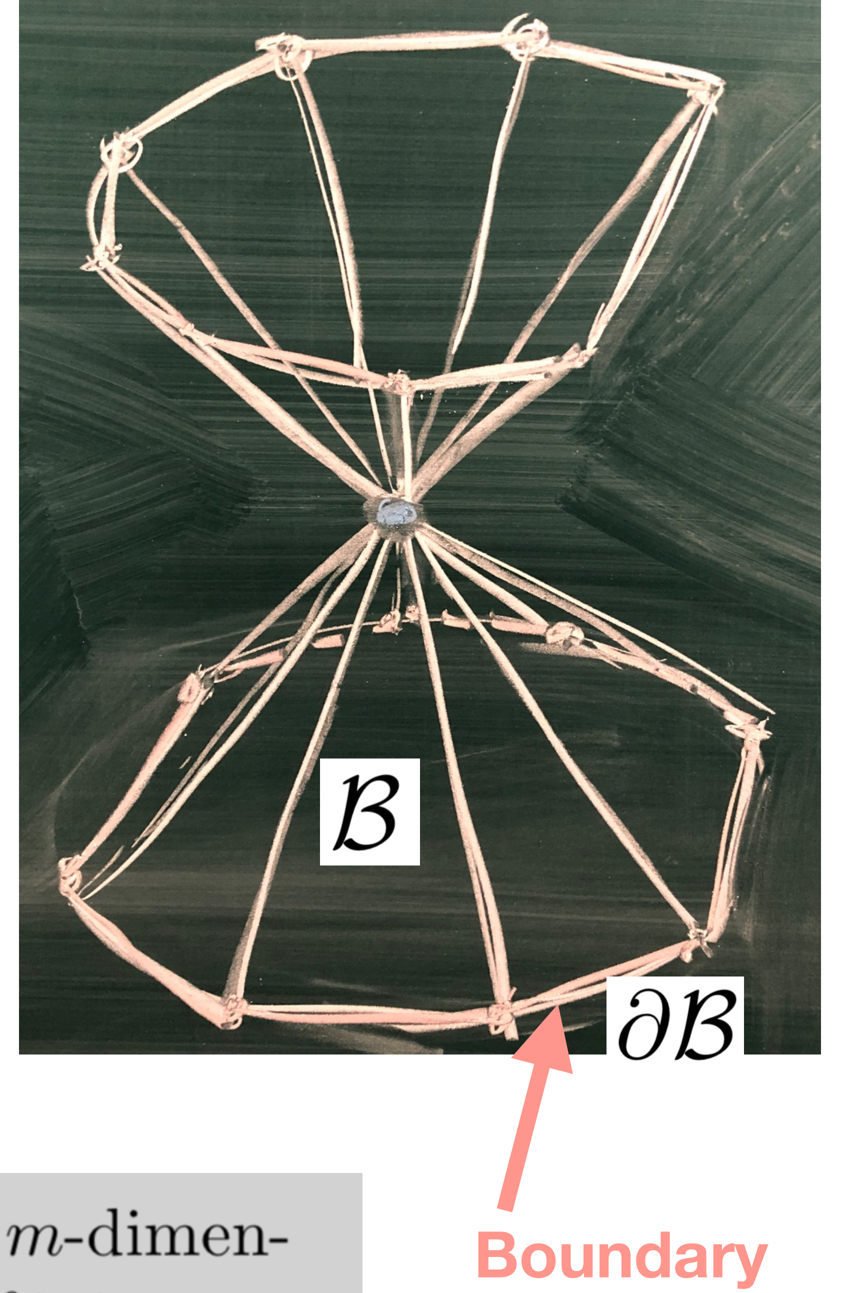
Pseudo-manifold: $n = 2$

Pseudo-manifold with boundaries: $1 \leq n \leq 2$

Definition 2. *A pure m-dimensional simplicial complex K is a* quasi-manifold *if any $(m-1)$-simplex has at least 2 full dimensional cofaces.*

Pseudo-manifolds

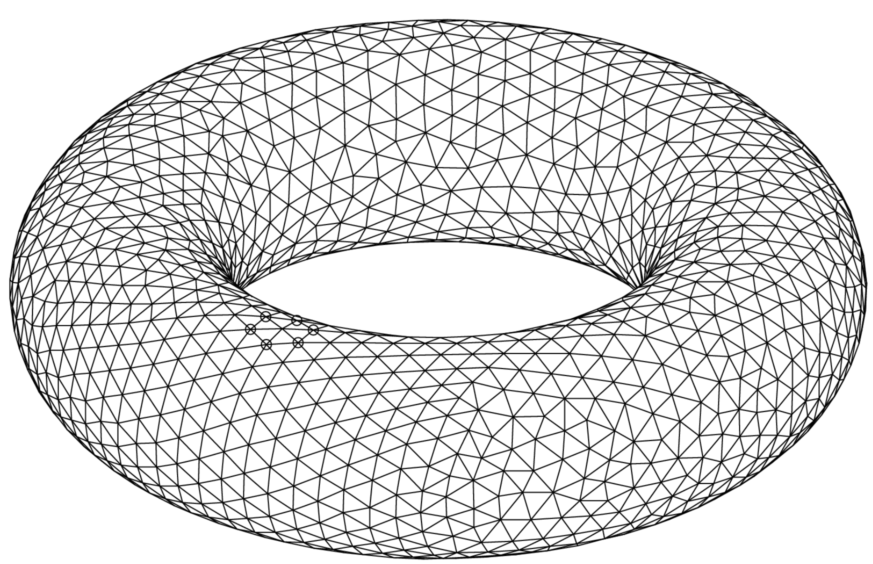


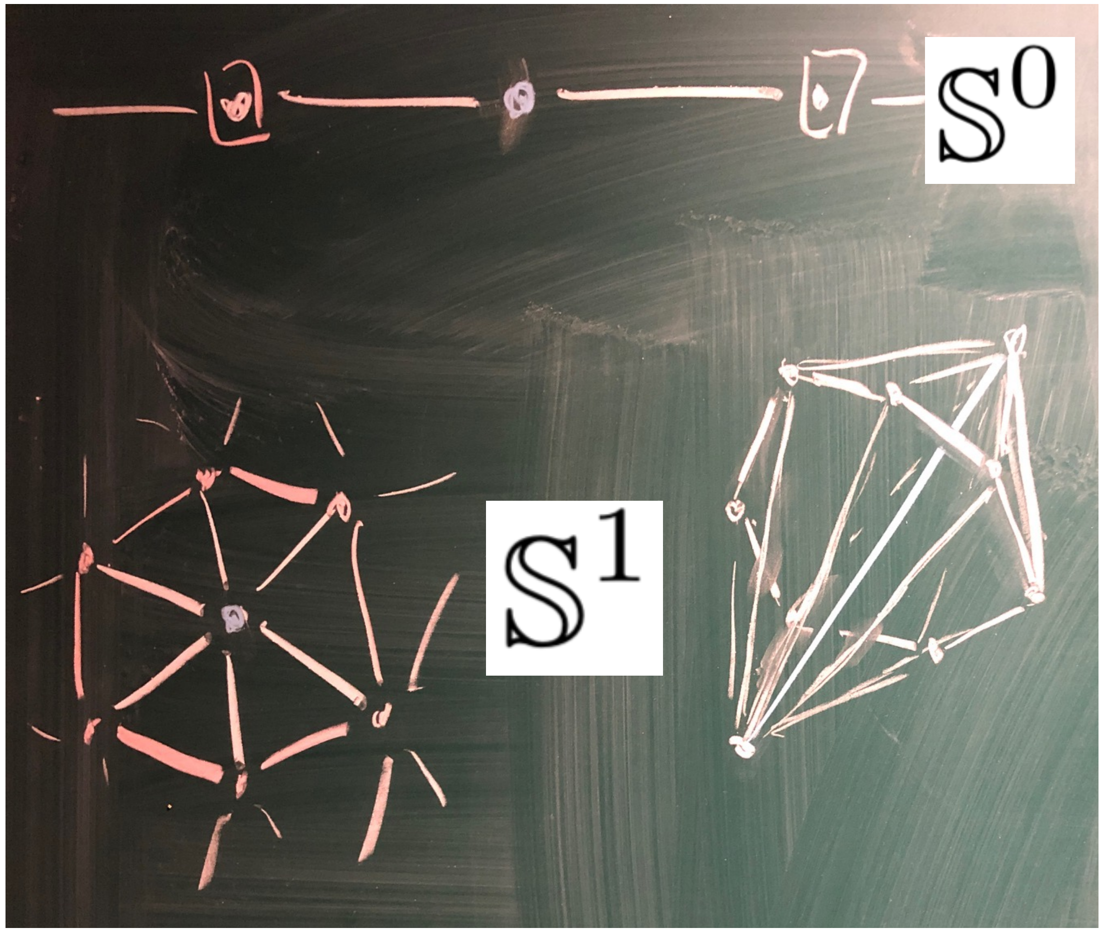


Definition 16 (Pseudo-manifold with boundary) An m-dimensional simplicial complex is a *pseudo-manifold with boundary* if it is pure and if any $(m-1)$-simplex has *at most two* m-dimensional cofaces. The *boundary* $\partial\mathcal{B}$ of an m-dimensional pseudo-manifold simplicial complex $\mathcal{B}$ with boundary, is the $(m-1)$-simplicial complex made of the closure of all $(m-1)$-simplices with exactly one m-dimensional coface, that is the simplices and their faces.

Manifolds

Definition 1. *A m-dimensional simplicial complex K is a* combinatorial manifold *if the link of each k-simplex is homeomorphic to the $(m-k-1)$-sphere $\mathbb{S}^{m-k-1}$.*

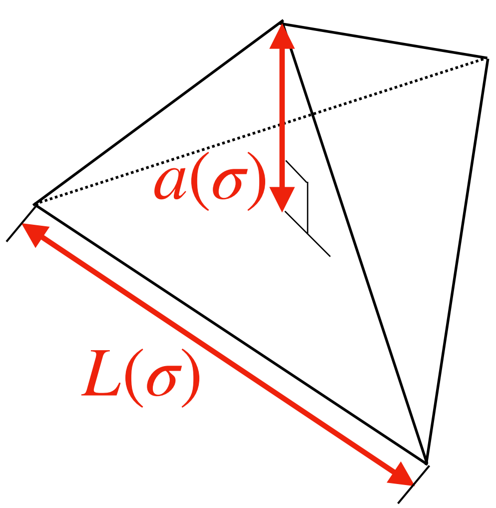




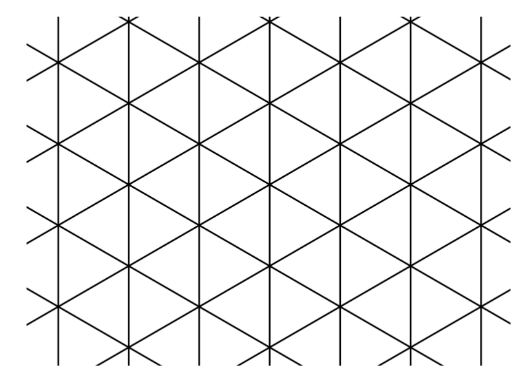
simplex quality

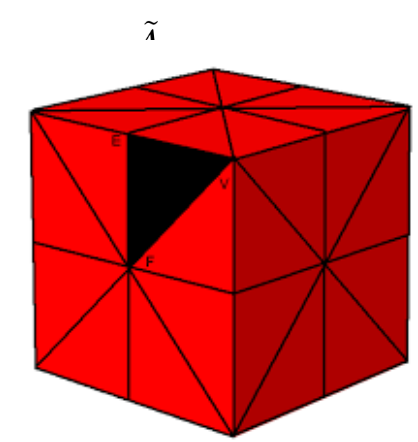
Not too large, not too flat:

Notation 1 (Simplex quality) The *thickness* of a m-simplex $\boldsymbol{\sigma}$, denoted $t(\boldsymbol{\sigma})$, is given by $\frac{a}{mL}$, where $a = a(\boldsymbol{\sigma})$ is the smallest altitude of $\boldsymbol{\sigma}$ and $L = L(\boldsymbol{\sigma})$ is the length of the longest edge. The altitude of a vertex in a simplex is the distance from the vertex to the affine hull of the opposite face. Observe that $t \leq 1/m$ and might be in fact $O(m^{-3/2})$ [CKW17]. We set $t(\boldsymbol{\sigma}) = 1$ if $\boldsymbol{\sigma}$ has dimension 0.



$$t(\sigma) = \frac{a(\sigma)}{mL(\sigma)}$$





Main theorem

Theorem 4 (Triangulation of submanifolds) *Let $M \subset \mathbb{R}^N$ be a connected C^2 m-dimensional submanifold of $\mathbb{R}^N$ with reach $\mathrm{rch}(M) > 0$, and $\mathcal{P} \subset M$ a finite set of points. Suppose that $\mathcal{A}$ is an m-dimensional quasi-manifold simplicial complex whose vertex set $\mathcal{A}^0$ is identified with $\mathcal{P}$. Let $L, t > 0$ be such that for any m-simplex $\boldsymbol{\sigma} \in \mathcal{A}$ one has:*

$$t \le t(\boldsymbol{\sigma}) \quad \text{and} \quad L(\boldsymbol{\sigma}) \le L.$$

If:

(a) *All simplices are small with respect to the reach:*

$$\frac{L}{\mathrm{rch}(M)} \le \min\left(\frac{1}{8}, t \sin \pi/8\right).$$

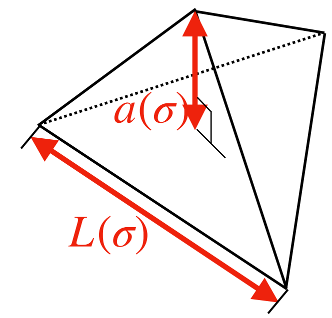
(b) *The projection of simplices on local tangent planes have disjoint interiors:*

$$\forall p \in \mathcal{P},\ \forall \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2 \in \mathcal{A} \text{ with } |\boldsymbol{\sigma}_1|, |\boldsymbol{\sigma}_2| \subset B(p, 2.8L),$$
$$\boldsymbol{\sigma}_1 \neq \boldsymbol{\sigma}_2 \Rightarrow \mathrm{pr}_{T_pM}(|\boldsymbol{\sigma}_1|)^\circ \cap \mathrm{pr}_{T_pM}(|\boldsymbol{\sigma}_2|)^\circ = \emptyset.$$

Then:

(1) *The inclusion $\iota : |\mathcal{A}| \to \mathbb{R}^N$ is an embedding, and we can identify $\iota(|\mathcal{A}|)$ with $|\mathcal{A}|$.*

(2) *The closest-point projection map $\mathrm{pr}_M|_{|\mathcal{A}|} : |\mathcal{A}| \to M$ is a homeomorphism, so M is compact, and there is an ambient isotopy bringing $|\mathcal{A}|$ to M.*



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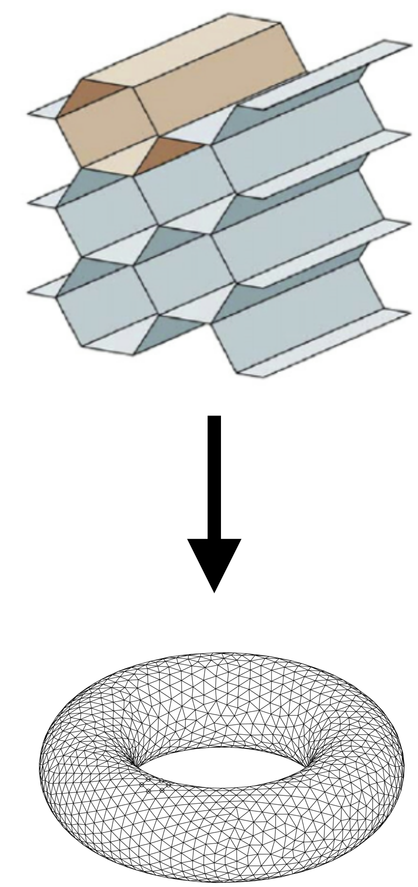
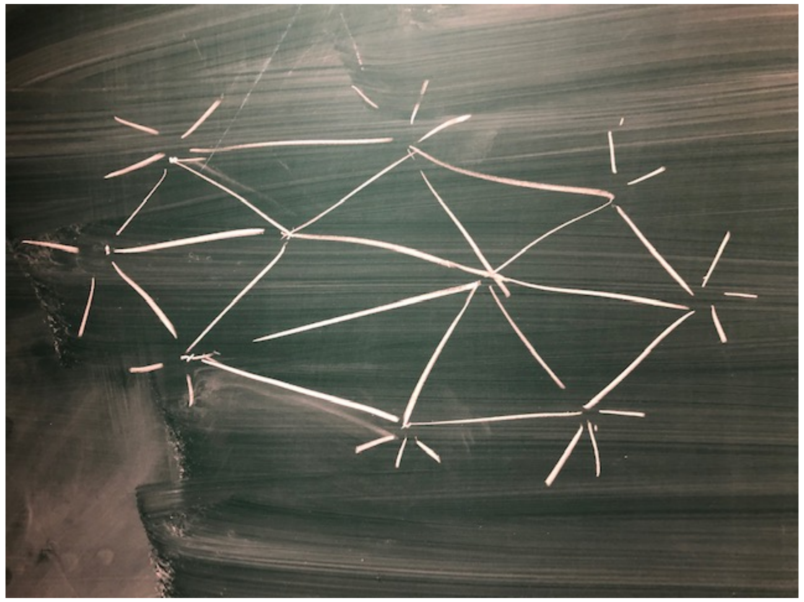


Euclidean version

K^m **is a** $m -$ **quasi-manifold** (i.e. Each *(m-1)*-simplex has at least two *m*-cofaces, i.e. « no boundaries »)

$\pi : K^m \to \mathbb{R}^m$ **is Piecewise Linear and non degenerate**

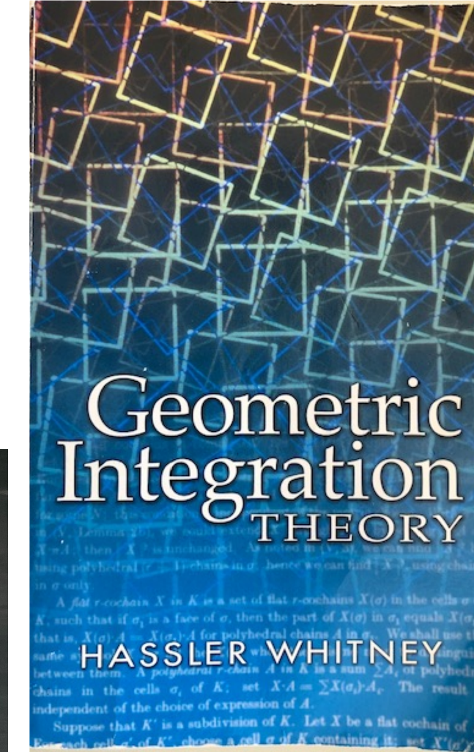
$$\sigma_1^m \neq \sigma_2^m \Rightarrow \pi(\sigma_1^m)^\circ \cap \pi(\sigma_2^m)^\circ = \varnothing$$



Then π **is injective and open and** K^m **is a manifold**

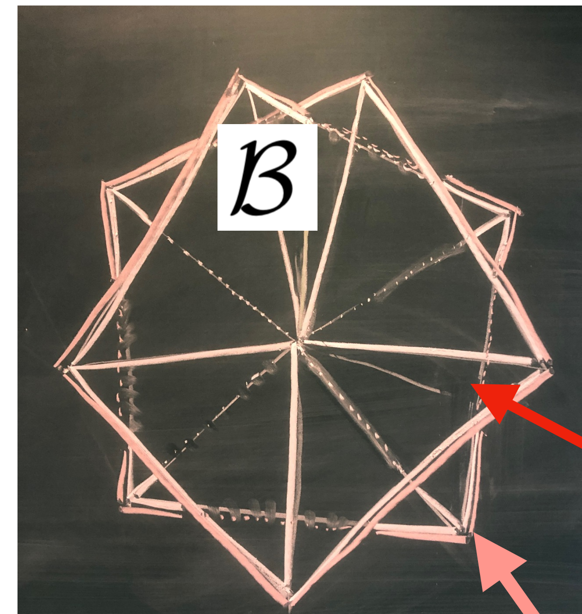
The proof of the euclidean case is a crucial step in the proof of the general Theorem:
it applies to the projection of a local subset of the simplicial complex a local tangent plane

Whitney Lemma



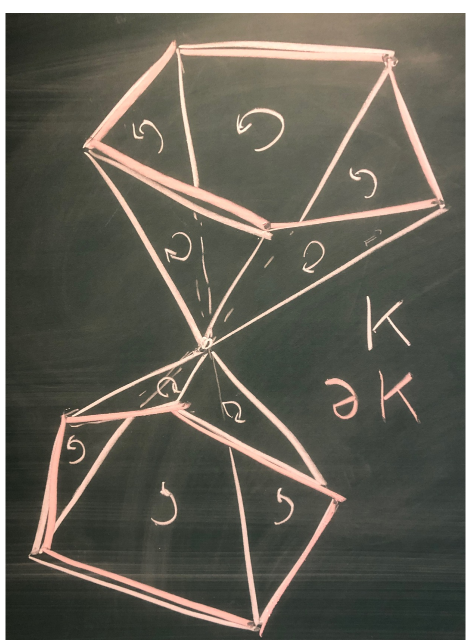
Definition 18 (Oriented pseudo-manifold) An m-dimensional pseudo-manifold with boundary $\mathcal{B}$ is said to be *oriented* if each m-simplex is given an orientation such that, if Γ is the m-chain over $\mathbb{Z}$ (or $\mathbb{R}$) with coefficient 1 on each m-simplex of $\mathcal{B}$ then the support of $\partial\Gamma$ is precisely $\partial\mathcal{B}$.

Definition 19 (Simplexwise positive map) Let $\mathcal{B}$ be an oriented m-dimensional pseudo-manifold with boundary. A piecewise linear map $F : \mathcal{B} \to \mathbb{R}^m$ is said to be *simplexwise positive* if the image $F(\sigma) = [F(v_0), \ldots, F(v_m)]$ of each oriented m-simplex $\sigma = [v_0, \ldots, v_m] \in \mathcal{B}$ is a non-degenerate m-simplex embedded in $\mathbb{R}^m$ and is positively oriented.



Simplexwise positive map

$\partial\mathcal{B}$



Oriented pseudo-manifolds

Whitney Lemma

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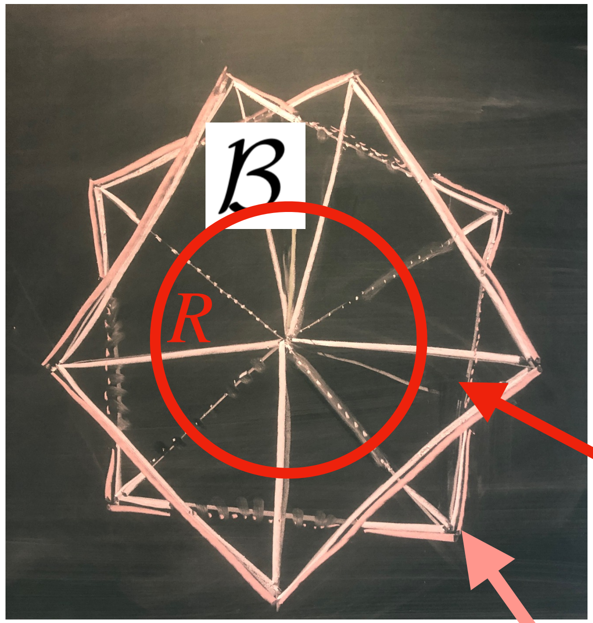

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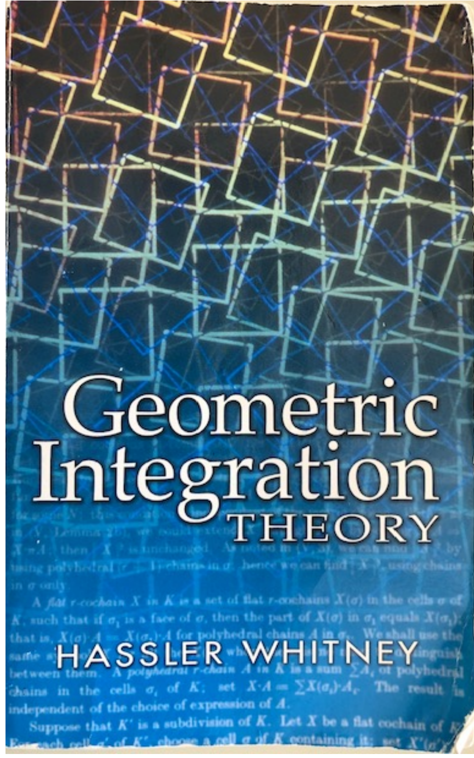


Lemma 2 (Whitney). *Assume that the following conditions are satisfied:*

(C1) *K is an oriented m-pseudo-manifold with boundary and $F : |K| \to \mathbb{R}^m$ is a simplexwise positive map.*

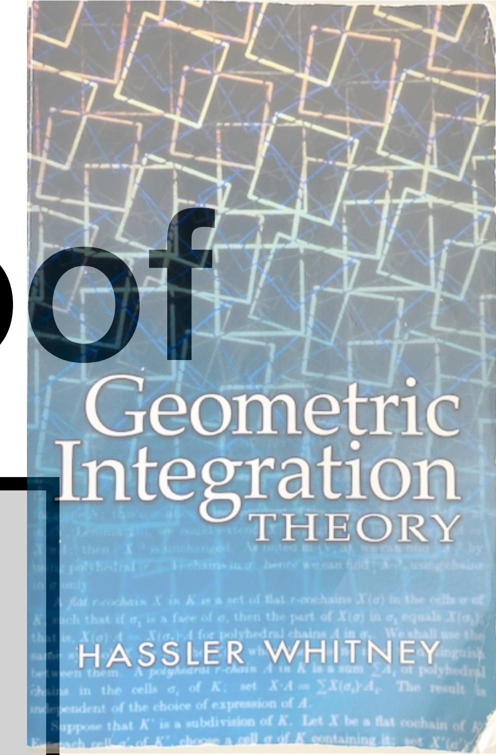
(C2) *$R \subset \mathbb{R}^m$ is a connected open set such that $R \cap F(|\partial K|) = \emptyset$.*

(C3) *There is a $y \in R \setminus F(|K^{m-1}|)$ such that $F^{-1}(y)$ is a single point.*

Then the restriction of F to $F^{-1}(R)$ is a bijection between $F^{-1}(R)$ and R.

C1+C2 => constant number of inverse images in R

Whitney Lemma proof



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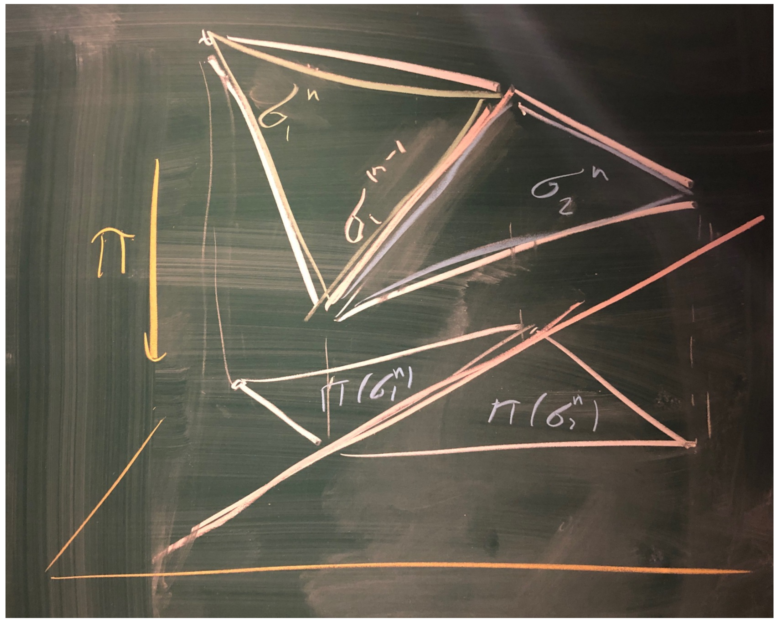
Whitney's proof in 4 steps:

1 $\forall i, \sigma_i^{m-1} \notin \partial K, \quad F_{|\text{star}\, \sigma_i^{m-1}}$ **is injective and open**

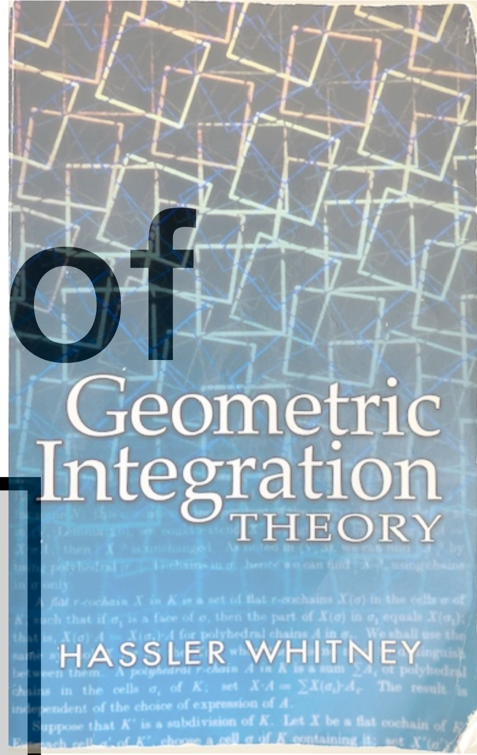


F **Simplexwise positive:**

$\Rightarrow$ **if** σ_1^m **and** σ_2^m **are the two full dimensional cofaces of** σ_i^{m-1}

$F(\sigma_1^m)$ **and** $F(\sigma_2^m)$ **are on opposite side of the hyper-plane spanned by** $F(\sigma_i^{m-1})$

Whitney Lemma proof



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Whitney's proof in 4 steps: Degree argument

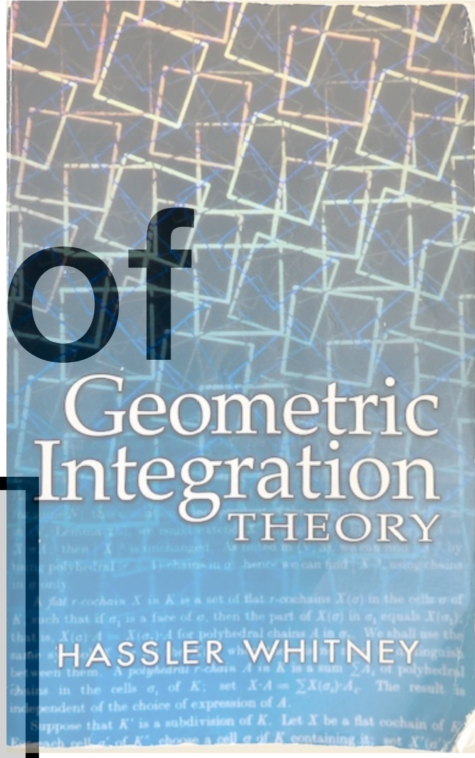
2 $x, y \in R \backslash F\left(K^{m-1}\right) \Rightarrow \#F^{-1}(x) = \#F^{-1}(y)$

$R \backslash F(K^{m-2})$ **is pathwise connected**

γ **is a path in** $R \backslash F(K^{m-2})$ **from** x **to** y

consider t_0 **such that** $t \mapsto \#F^{-1}\gamma(t)$ **changes at** t_0

Whitney Lemma proof



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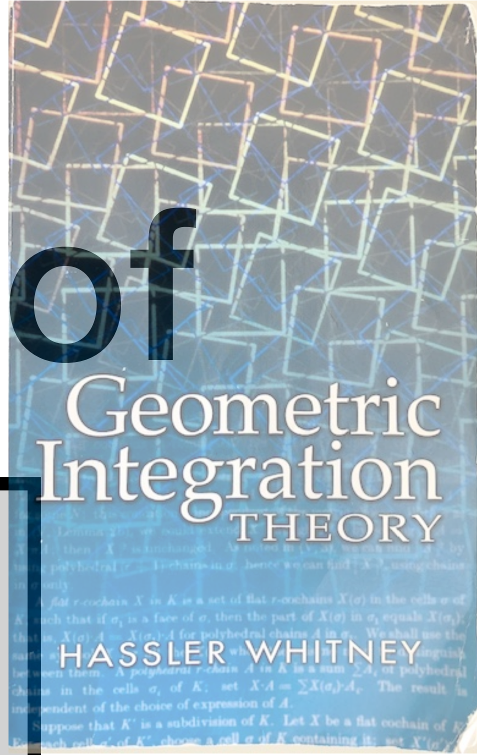

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C1+C2 => F is open

Whitney's proof in 4 steps:

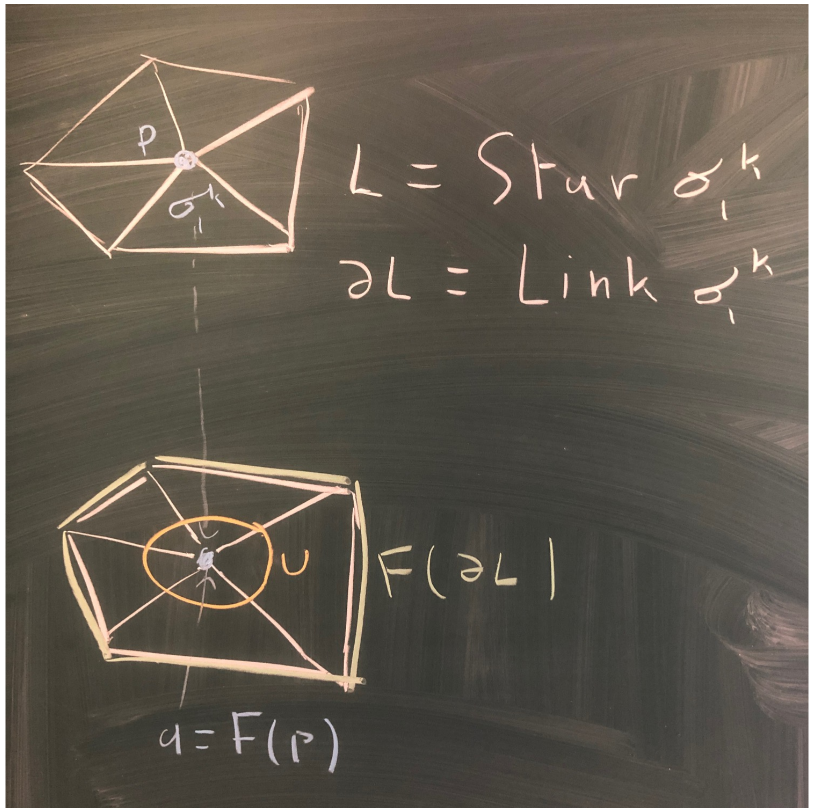
3 $\forall k, i, \sigma_i^k \notin \partial K, \quad F(\mathrm{star}\, \sigma_i^k)$ **is open and** $F_{|\mathrm{star}\, \sigma_i^k}$ **is open**

Consider $p \in \mathrm{int}(\sigma_i^k)$ **and** U **a neighborhood of** $q = F(p)$

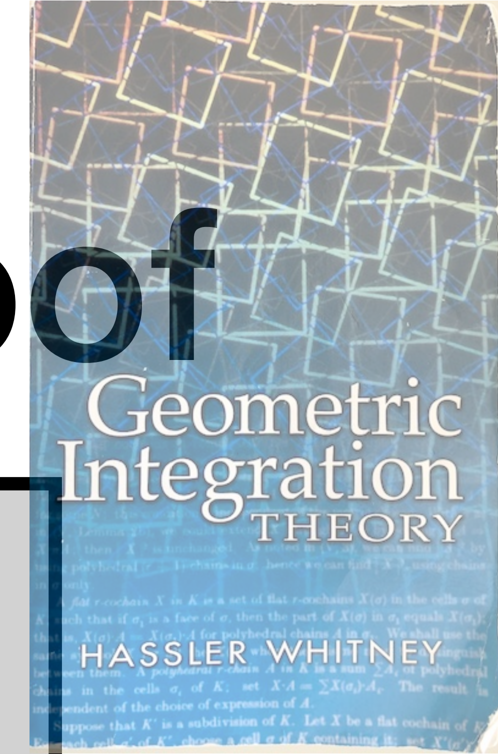
Defines: $L = \overline{\mathrm{star}\, \sigma_i^k}$ **and apply step (2) to** L :

if $U \cap \partial L = \varnothing$ **each point in** $U \backslash F(L^{m-1})$**is covered**

since $U \subset \overline{U \backslash F(L^{m-1})}$ **we have** $U \subset F(L)$



Whitney Lemma proof



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Whitney's proof in 4 steps:

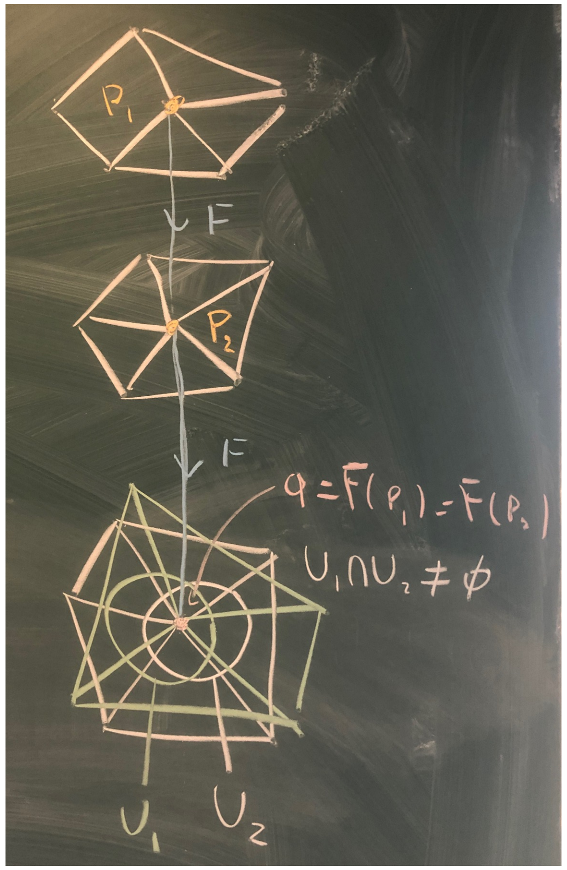
4 $\exists x \in R \backslash F\left(K^{m-1}\right), \#F^{-1}(x) = 1 \Rightarrow F_{|f^{-1}(R)}$ **is injective:**

Consider $p_1 \neq p_2, F(p_1) = F(p_2) = q$ **with** $p_i \in \text{int}(\sigma_i), i = 1{,}2$

F **is one-to-one on simplices** $\Rightarrow \text{star}\,\sigma_1 \cap \text{star}\,\sigma_2 = \varnothing$

step (3) $\Rightarrow F(\text{star}\,\sigma_i)$ **covers some open set** $U_i \ni q$

$U_1 \cap U_2$ **is an open set covered at least twice**



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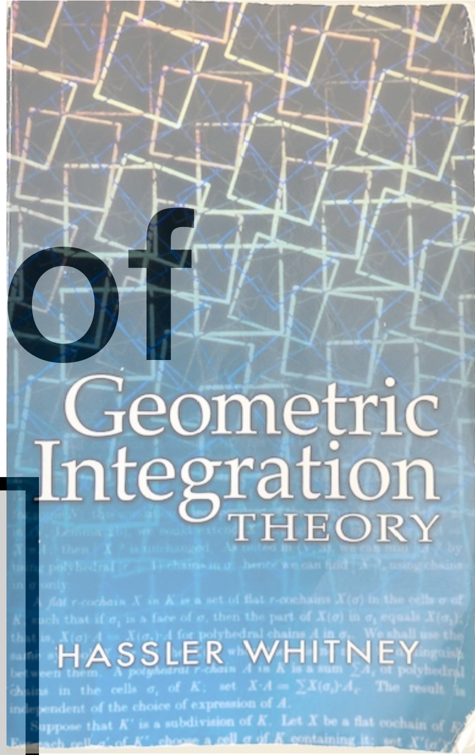

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Consequence of Whitney's Theorem:

F **is injective and open, and is therefore an homeomorphism on it image**

An alternative to Whitney Lemma: recursion on the dimension

(*after some discussions with* **David Cohen-Steiner**)

Theorem 3. *Let $d \geq 0$ and $\mathcal{K}$ be a finite, d-dimensional simplicial complex. For a piecewise linear map $\phi : |\mathcal{K}| \to \mathbb{R}^{d+1} \setminus \{0\}$ define $\mathring{\phi} : |\mathcal{K}| \to \mathbb{S}^d$ as :*

$$\mathring{\phi}(x) = \frac{\phi(x)}{\|\phi(x)\|}$$

If $\mathcal{K}$ and $\mathring{\phi}$ satisfy:

1. *Any simplex $\tau \in \mathcal{K}$ with $\dim \tau \leq d-1$ has at least two d-dimensional cofaces for $d \geq 1$, and $\mathcal{K}$ has at least two 0-dimensional simplices for $d = 0$.*
2. *For any $\sigma \in \mathcal{K}_d$, the restriction of $\mathring{\phi}$ to $|\sigma|$ is injective.*
3. *The restriction of $\mathring{\phi}$ to*
$$\bigcup_{\sigma \in \mathcal{K}_d} \sigma^\circ$$
is injective.

Then $\mathring{\phi}$ is a homeomorphism between $|\mathcal{K}|$ and $\mathbb{S}^d$.

Unfortunately it is not simpler (formal proof = 7 pages)

Back to the theorem

Theorem 4 (Triangulation of submanifolds) *Let $M \subset \mathbb{R}^N$ be a connected C^2 m-dimensional submanifold of $\mathbb{R}^N$ with reach $\mathrm{rch}(M) > 0$, and $\mathcal{P} \subset M$ a finite set of points. Suppose that $\mathcal{A}$ is an m-dimensional quasi-manifold simplicial complex whose vertex set $\mathcal{A}^0$ is identified with $\mathcal{P}$. Let $L, t > 0$ be such that for any m-simplex $\boldsymbol{\sigma} \in \mathcal{A}$ one has:*

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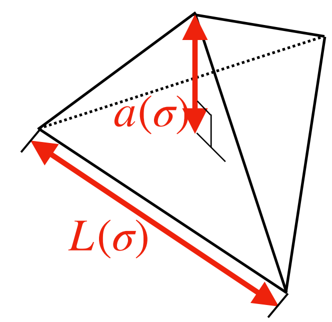

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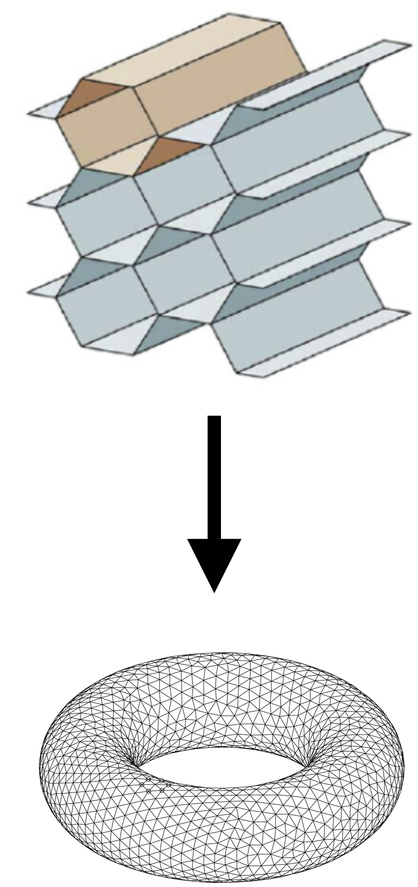

Invariance of Domain Theorem

*(A kind of **inverse function theorem** for **continuous** maps)*

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Invariance of domain is a theorem in topology about homeomorphic subsets of Euclidean space $\mathbf{R}^n$. It states:

> If U is an open subset of $\mathbf{R}^n$ and $f : U \to \mathbf{R}^n$ is an injective continuous map, then $V = f(U)$ is open and f is a homeomorphism between U and V.

The theorem and its proof are due to L. E. J. Brouwer, published in 1912.[1] The proof uses tools of algebraic topology, notably the Brouwer fixed point theorem.

[Bro12] L.E.J. Brouwer. Beweis der Invarianz des n-dimensionalen Gebiets. *Mathematische Annalen*, 71:305–313, 1912. 9

General condition for homeomorphism

Theorem 3 (Triangulation of manifolds)**.** *Let H be a continuous mapping from a (non-empty) m-dimensional finite simplicial complex $\mathcal{A}$, to a connected m-manifold without boundary M.*

If H is injective and $\mathcal{A}$ is a manifold, then H is a homeomorphism.

Proof By the invariance of domain theorem [Bro12], we have that H is open. Being injective, continuous and open, H is a homeomorphism on its image. Since $\mathcal{A}$ is finite, it is compact and $H(|\mathcal{A}|)$ is the image of an open and compact set by an open and continuous map and is therefore open and compact. Since M is connected its only open and closed non-empty subset is M itself, therefore $H(|\mathcal{A}|) = M$. □

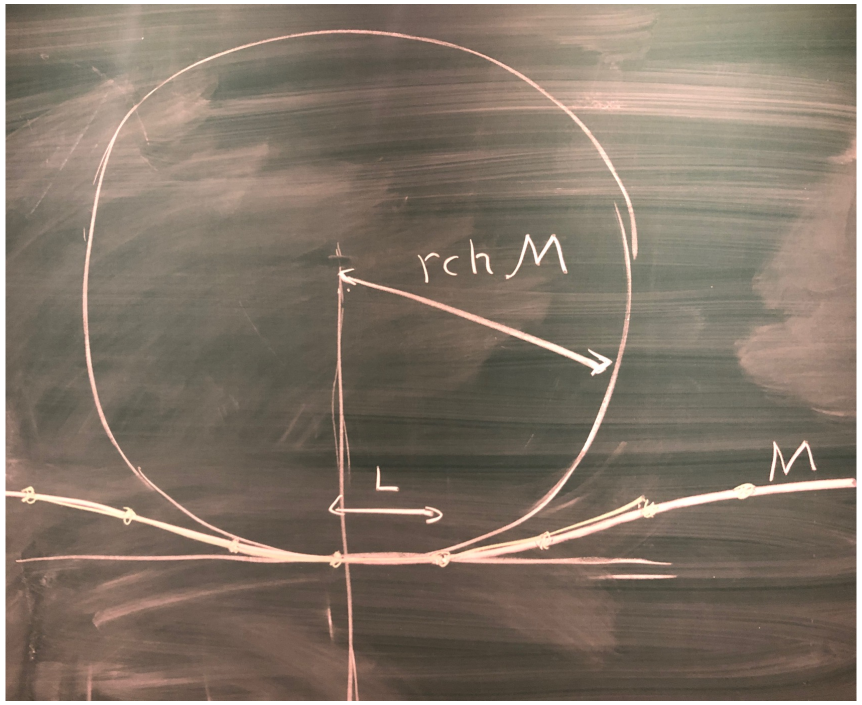
Proof outline

First some geometry:

« Simplices are not too flat (t) and small (L) with respect to reach of M »

Then:

$$\frac{L}{\mathrm{rch}(M)} \leq \min\left(\frac{1}{8}, t\, \sin \pi/8\right)$$



Lemma 8 (Tangent Balls) *For any $p \in M$, any open ball $B(c, r)$ that is tangent to M at p and whose radius r satisfies $r \leq \mathrm{rch}(M)$ does not intersect M.*

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Lemma 2 (Whitney). *Assume that the following conditions are satisfied:*

(C1) *K is an oriented m-pseudo-manifold with boundary and $F : |K| \to \mathbb{R}^m$ is a simplexwise positive map.*

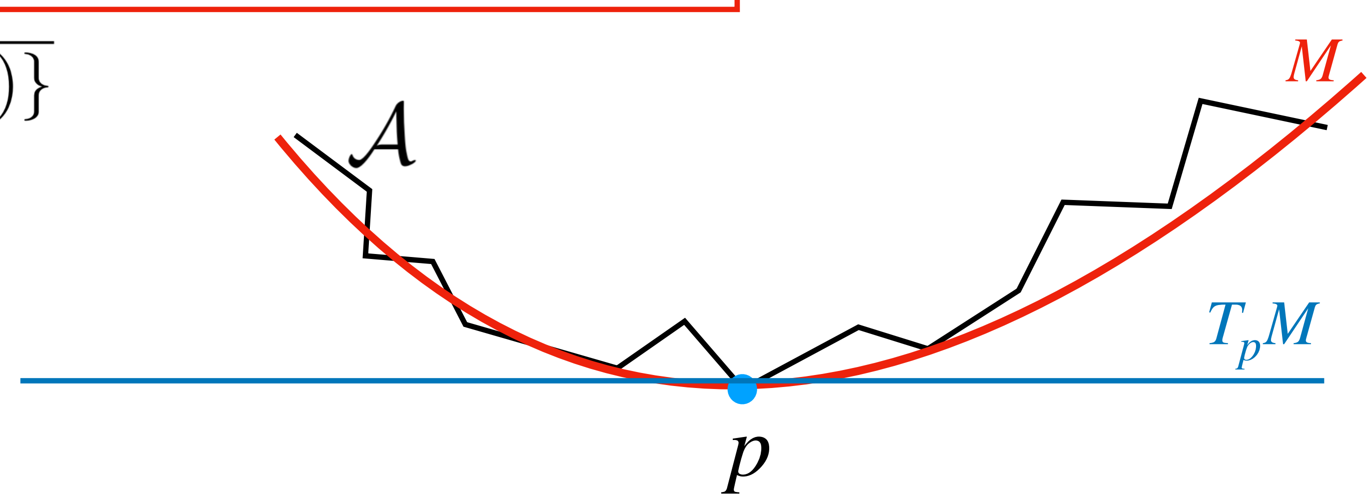
(C2) *$R \subset \mathbb{R}^m$ is a connected open set such that $R \cap F(|\partial K|) = \emptyset$.*

(C3) *There is a $y \in R \setminus F(|K^{m-1}|)$ such that $F^{-1}(y)$ is a single point.*

Then the restriction of F to $F^{-1}(R)$ is a bijection between $F^{-1}(R)$ and R.

1) Whitney Lemma applies locally to the projection on a local tangent plane

$$\mathcal{A}_{p,\rho} = \overline{\{\boldsymbol{\sigma} \in \mathcal{A} \mid \dim(\boldsymbol{\sigma}) = m,\ \boldsymbol{\sigma} \subset B(p,\rho)\}}$$



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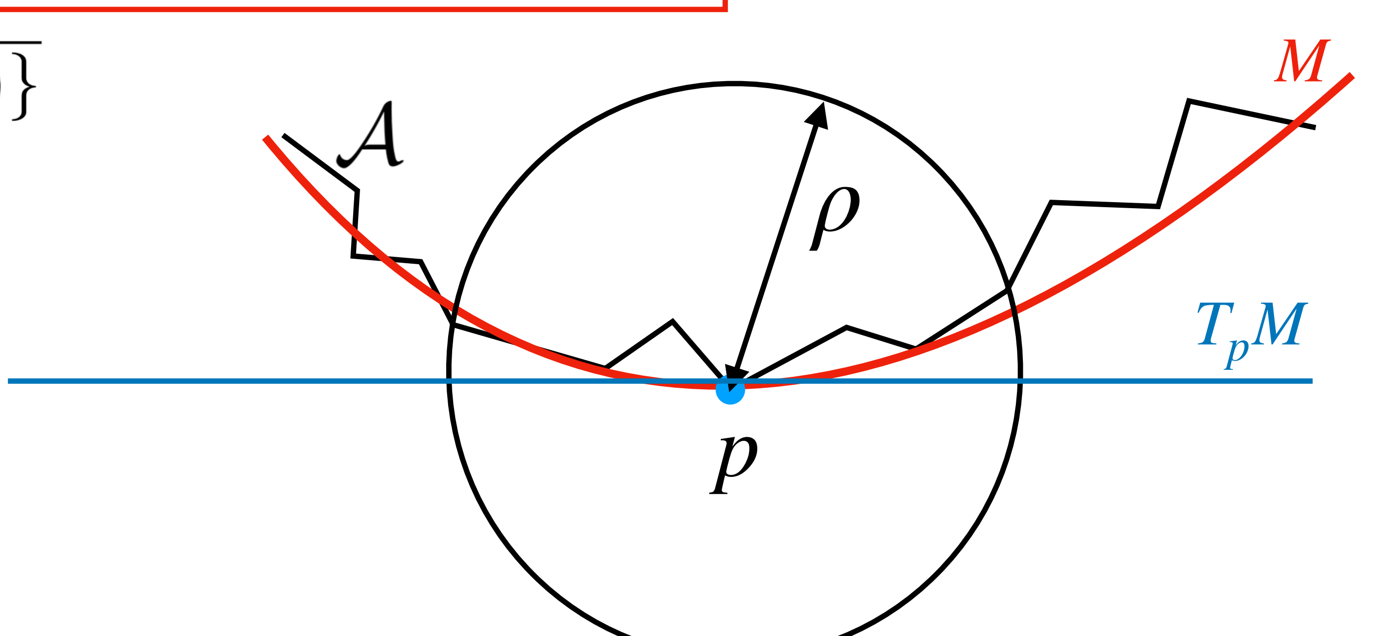

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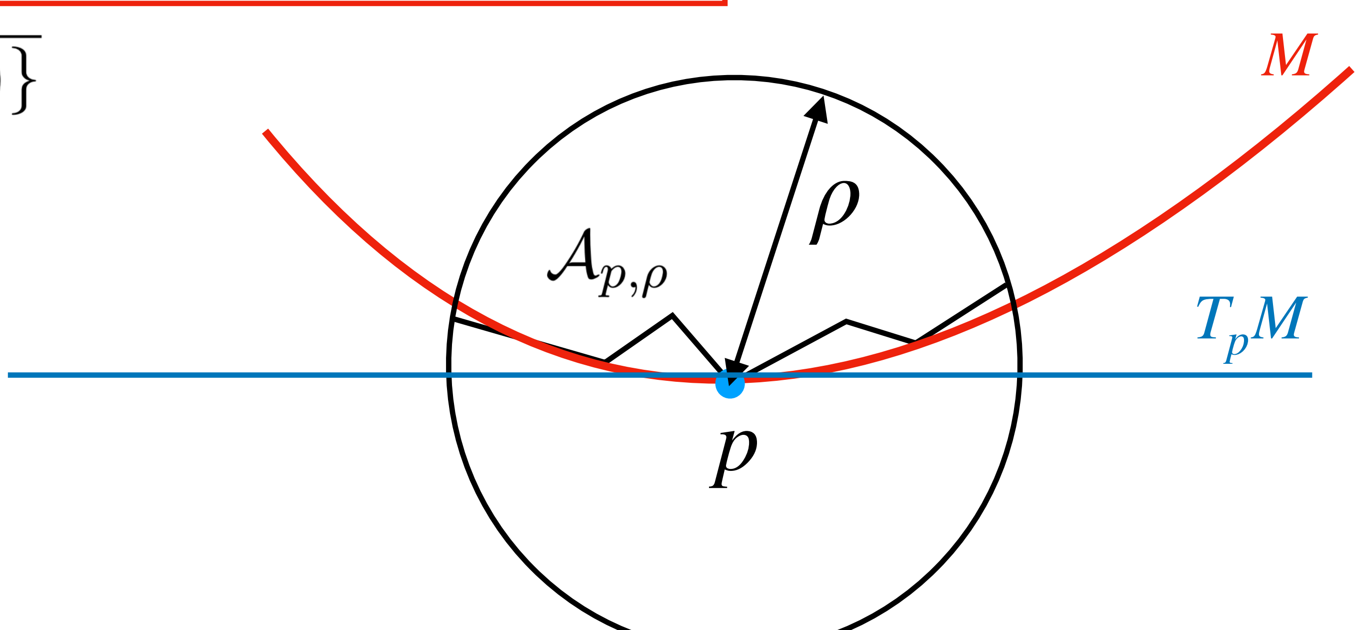

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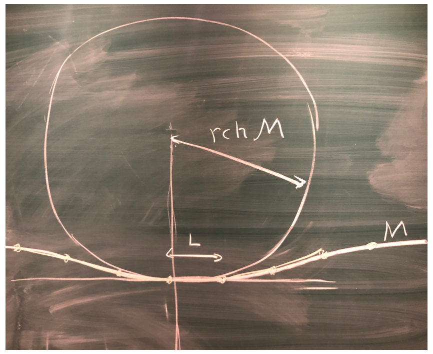

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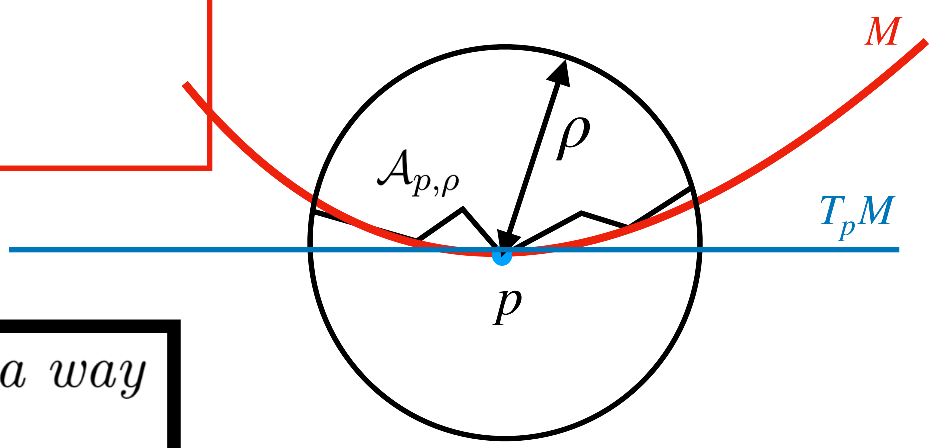

Claim 25 *The m-simplices of $\mathcal{A}_{p,2.8L}$ can be oriented in such a way that:*

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C1 Condition of Whitney'sLemma

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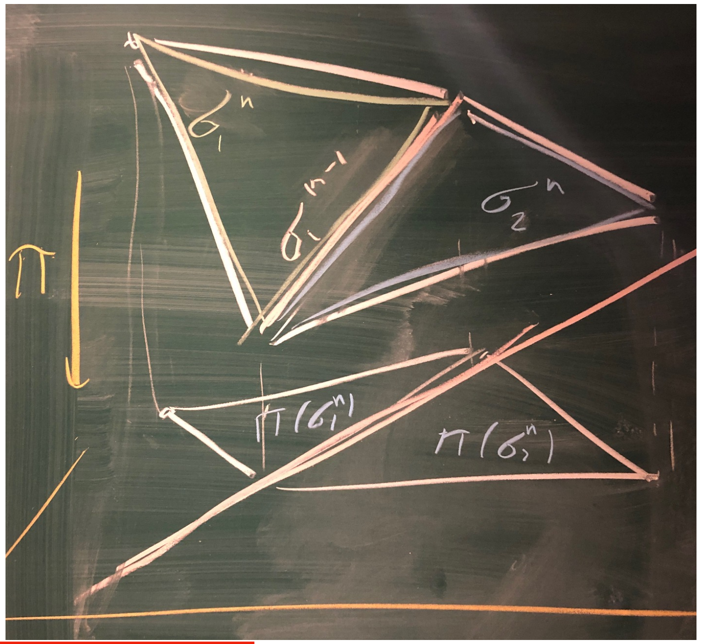

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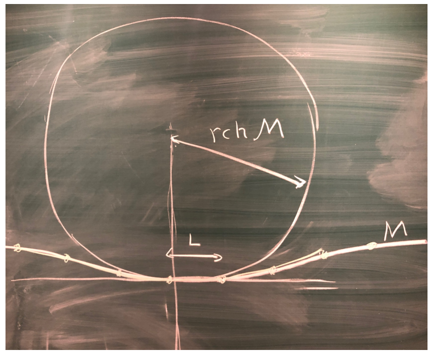

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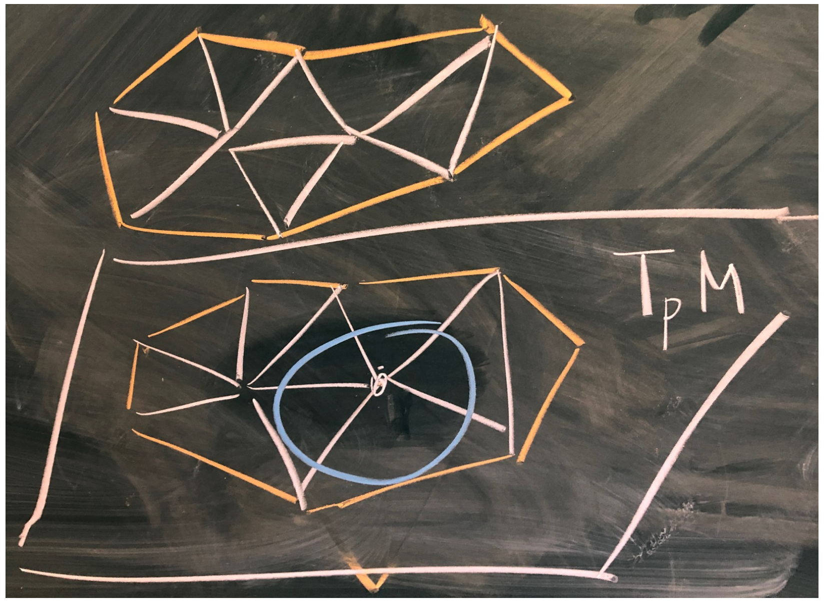
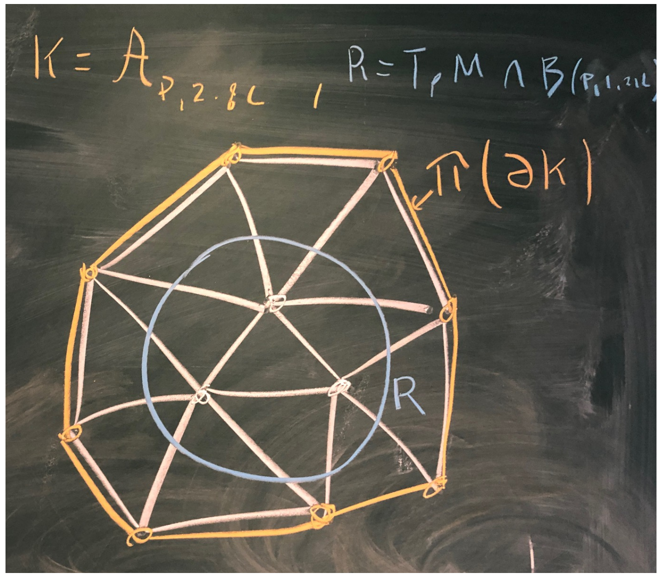
$$\widehat{\mathbf{R}}_p = T_pM \cap B(p, 1.21L)$$

$$\widehat{\mathbf{R}}_p \cap F_p(|\partial \mathcal{A}_{p,2.8L}|) = \emptyset$$

C2 Condition of Whitney'sLemma

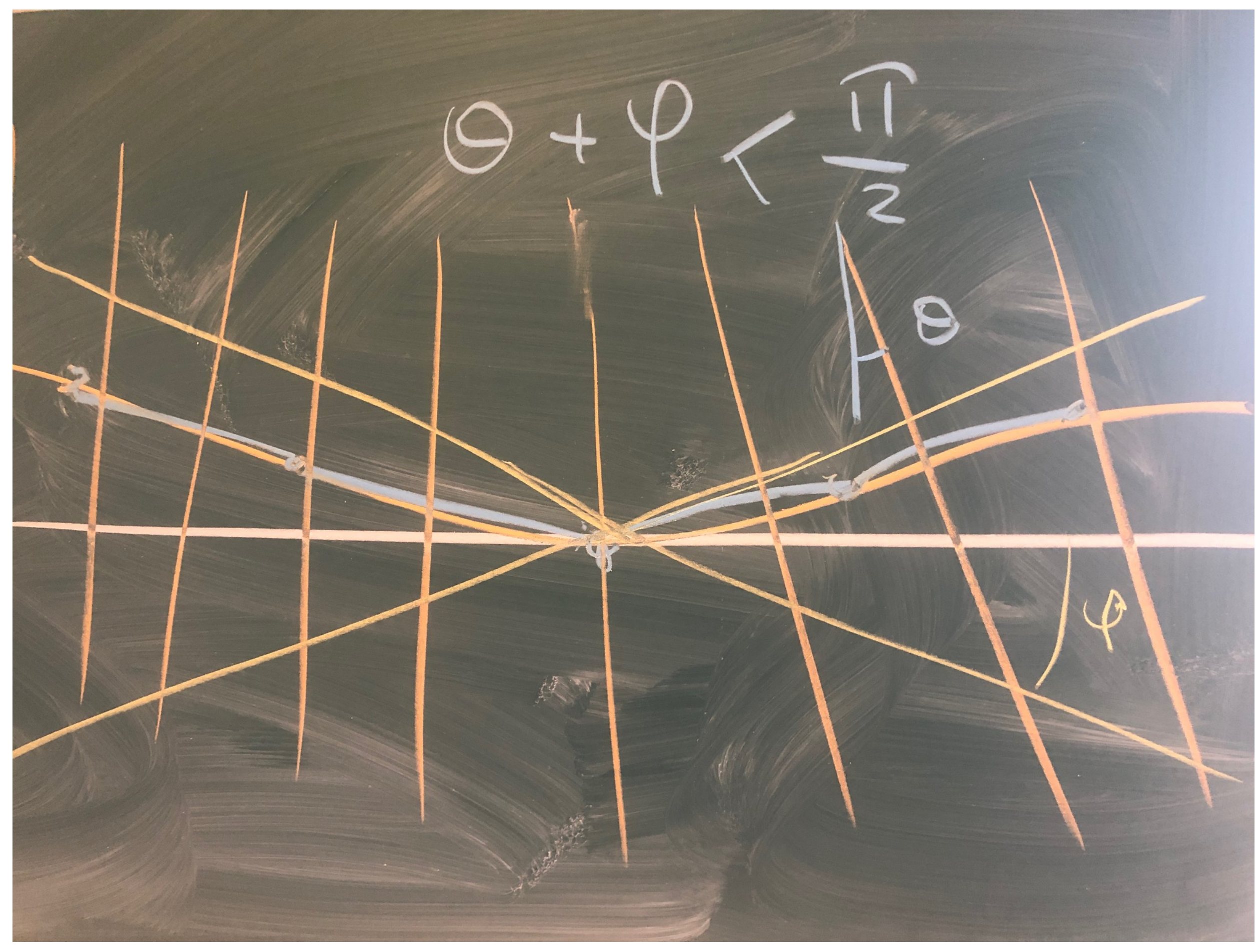
=> $\mathcal{A}$ is a manifold





Proof outline

By angles argument the projection on the manifold is locally injective.

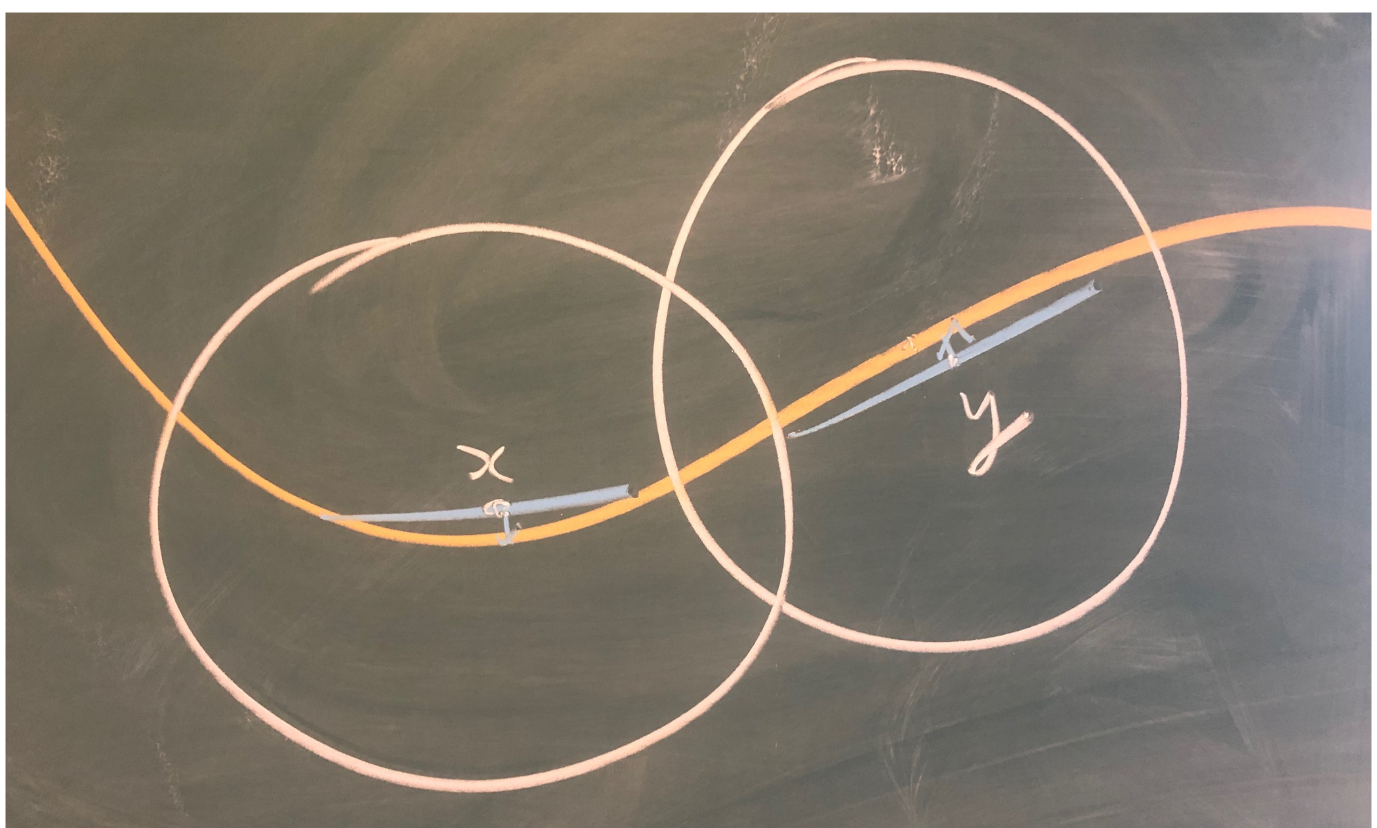


Proof outline

Since: $$x \in |\mathcal{A}| \Rightarrow \|x - pr_M(x)\| < \frac{L}{4}$$

$$\|x - y\| > \frac{L}{2} \Rightarrow pr_M(x) \neq pr_M(y)$$

$pr_M\big|_{|\mathcal{A}|}$ is globally injective



Thank you !