Triangulating submanifolds: An elementary and quantified version of Whitneys method

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Triangulating manifolds

A short subjective history

- Cairns (1934)
  - Classical, abstract, difficult result
- Whitney (1956)
  - Simpler embedding result
- Closed ball property [Edelsbrunner, Shah] (1994)
  - Formulated in $\mathbb{R}^d$ (can be fixed)
  - Delaunay of input set
- Amenta and Bern (late 1990s)
  - Voronoi of input set based (surfaces in $\mathbb{R}^3$)
- Dey e.a. (early 2000s)
  - Delaunay of input set, exponential in d
- Riemannian barycentric coordinates [Dyer, Vegter, W.] (2015)
  - Abstract differentiable manifold
- Tangential Delaunay complex [B., Ghosh] (2009)
  - Submanifolds of $\mathbb{R}^d$

Whitney's method stands out because of the following reasons:

- Historic importance in mathematics
- Not studied in the CG community
- Ambient triangulation, not Delaunay based
- Similarities with isosurfacing

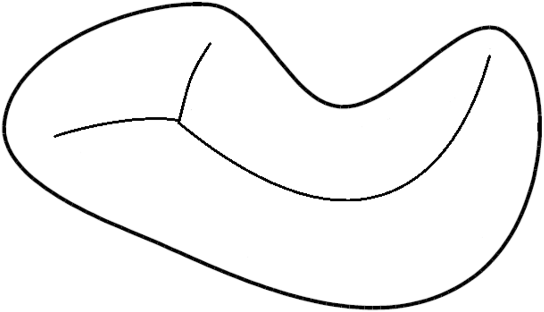
The reach (Federer 1959)

Setting: closed $\mathcal{S} \subset \mathbb{R}^n$

The medial axis ($\mathrm{ax}(\mathcal{S})$)

All points for which the closest point on $\mathcal{S}$ is NOT unique.

The reach of $\mathcal{S}$, denoted by $\mathrm{rch}(\mathcal{S})$ is the smallest distance from a point on the medial axis to $\mathcal{S}$

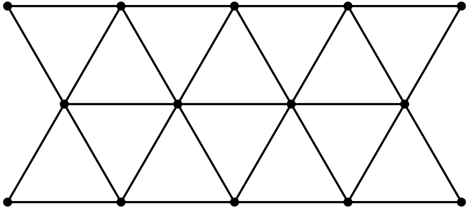


Coxeter triangulations

Definition

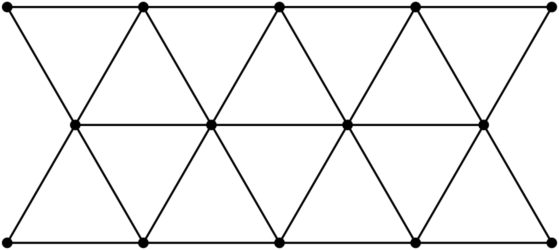
A monohedral[a] triangulation is called a *Coxeter triangulation* if all its d-simplices can be obtained by consecutive orthogonal reflections through facets of the d-simplices in the triangulation and all reflection planes are triangulated by facets in the triangulation.

[a] A triangulation of $\mathbb{R}^d$ is called *monohedral* if all its d-simplices are congruent.



Coxeter triangulations

All Coxeter triangulations have great quality.
There is one particular Coxeter triangulation of type $\tilde{A}^d$ in each dimension, which is also Delaunay protected, meaning it is a very stable Delaunay triangulation.
Very easy to store and efficient operations [PhD-thesis Siargey Kachanovich]



Goals paper

- Algorithmize the constructive triangulation proof by Whitney (first part of this presentation)
- Quantify all constants involved in terms of the reach of the C^2 manifold (technical)
- Find a new elementary homeomorphism proof, based on a tubular neighbourhood of the triangulation, which works simplex-wise (main focus presentation)
- Simplify other parts of the proof by the use of Coxeter triangulations of type $\tilde{A}^d$

The algorithm

The algorithm

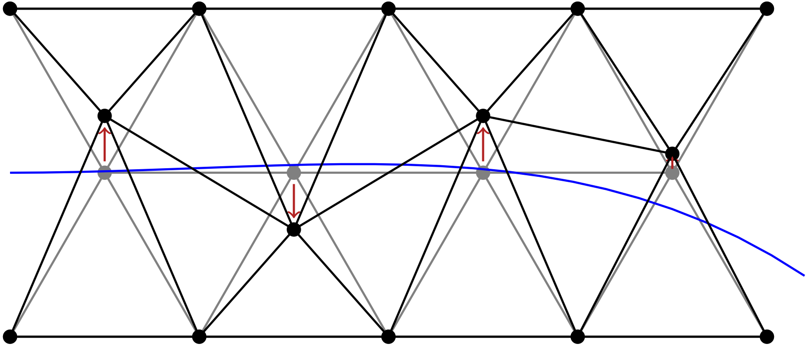
Input: An n-dimensional compact C^2 manifold $\mathcal{M}$ in $\mathbb{R}^d$ with reach $\mathrm{rch}(\mathcal{M})$
Output: A triangulation K of $\mathcal{M}$

Two parts:

- Take a sufficiently fine triangulation of type $\tilde{A}^d$ and perturb the vertices such that the intersection with every simplex τ is nice.
- Triangulate $\mathcal{M}$ by triangulating the intersection of $\mathcal{M}$ with τ.

The algorithm: part 1

Perturb the vertices a little bit such that there is lower bound on the distance between $\mathcal{M}$ and the $d - n - 1$-skeleton of the ambient triangulation.



Perturbing the vertices

- Start with an empty list of vertices and simplices.
- Add all vertices that are far from $\mathcal{M}$. Add the simplices such that the combinatorics of the $\tilde{A}^d$ Coxeter triangulation is preserved.
- For a vertex v_i close to $\mathcal{M}$, choose a $p \in \mathcal{M}$ nearby. Look at all τ' such that $\dim(\tau') \leq d - n - 2$ and $v_i * \tau'$ corresponds to a simplex of $\tilde{A}^d$ Coxeter triangulation. Perturb v_i such that it is sufficiently far from $\mathrm{aff}(T_p\mathcal{M}, \tau')$ for all such τ' and add the perturbed vertex to the list as well as all $v_i * \tau'$. (here bound on reach comes in)

(Note that because $\tilde{A}^d$ is a protected Delaunay triangulation, perturbing the vertices does not cause problems)

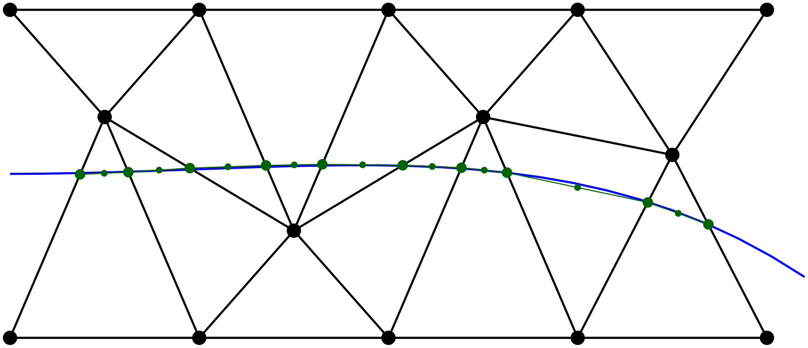
Geometric properties after intersection

Suppose that p is close to τ, then

- $T_p\mathcal{M}$ intersects τ if and only if $\mathcal{M}$ intersects τ
- M and τ^{d-n} intersect at most in a single point.
- There is a lower bound on the angle between $\mathrm{aff}(\tau^{d-n})$ and $T_p\mathcal{M}$, with τ^{d-n} a $(d-n)$-face in the ambient triangulation.

The algorithm: part 2

Defining the triangulation K of $\mathcal{M}$.



The algorithm: part 2

Roughly speaking; the intersection of $\mathcal{M}$ and τ is a polytope (will be proven later). The barycentric subdivisions of these τs give K.
Formally:
For $\tau_0 \subset \tau_1 \subset \cdots \subset \tau_k$ such that τ_0 intersects $\mathcal{M}$,

$$\sigma^k = \{v(\tau_0), \ldots, v(\tau_k)\}$$

will be a simplex of K.

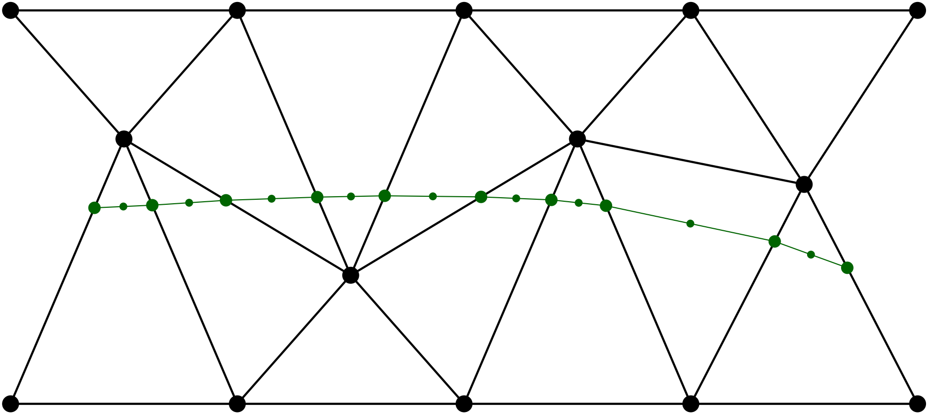
- If $\dim(\tau) = d - n$, then define $v(\tau) = \mathcal{M} \cap \tau$.
- If $\dim(\tau) \geq d - n$, consider the faces $\tau_1^{d-n}, \ldots, \tau_j^{d-n}$ of τ intersect $\mathcal{M}$. Define $v(\tau)$ as follows:

$$v(\tau) = \frac{1}{j}\left(v(\tau_1^{d-n}) + \cdots + v(\tau_j^{d-n})\right).$$

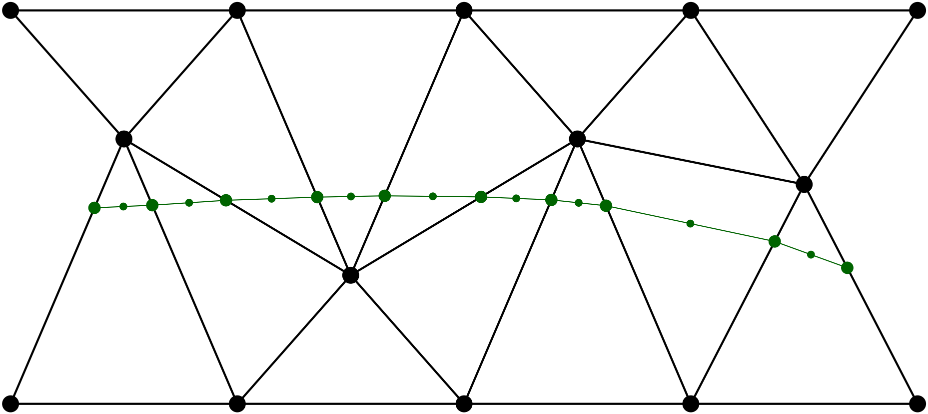
Homeomorphism proof: tubular neighbourhoods of K

Tubular neighbourhood of K

We will construct a tubular neighbourhood of K that is adapted to the ambient triangulation. We do so by indicating 'normal' spaces.

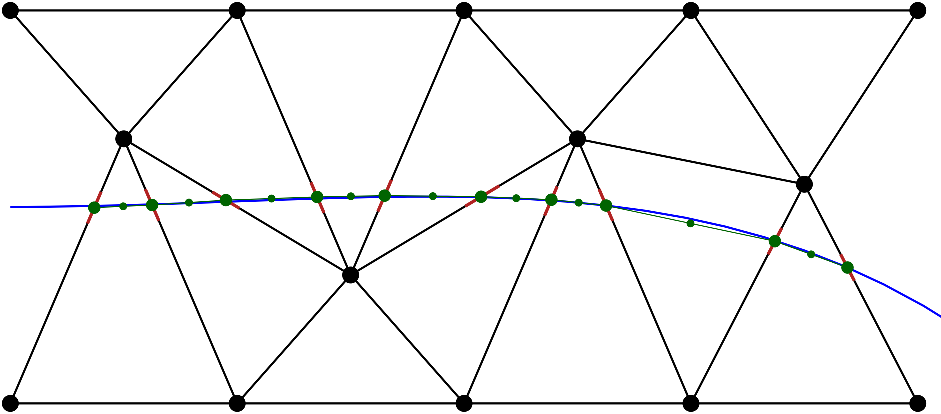


Remember simplices τ^{d-n} intersect $\mathcal{M}$ in at most one point and at these points we have vertices of our triangulation.



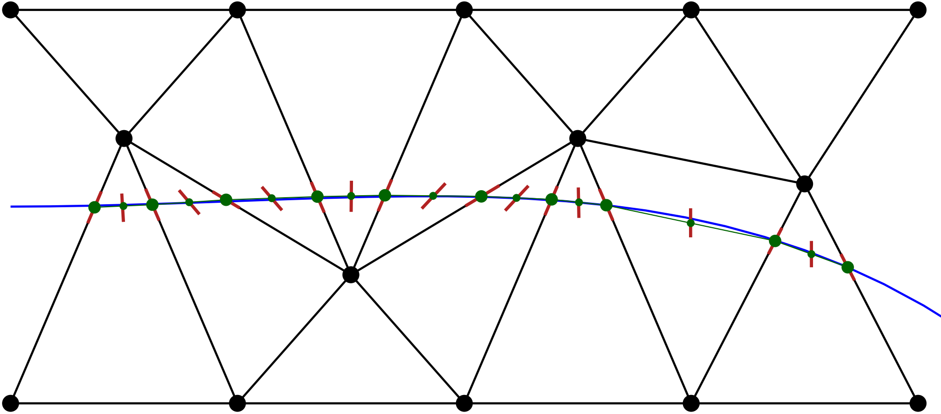
Tubular Neighbourhoods

At the point of intersection between τ^{d-n} and $\mathcal{M}$, we choose $\mathrm{aff}(\tau^{d-n})$



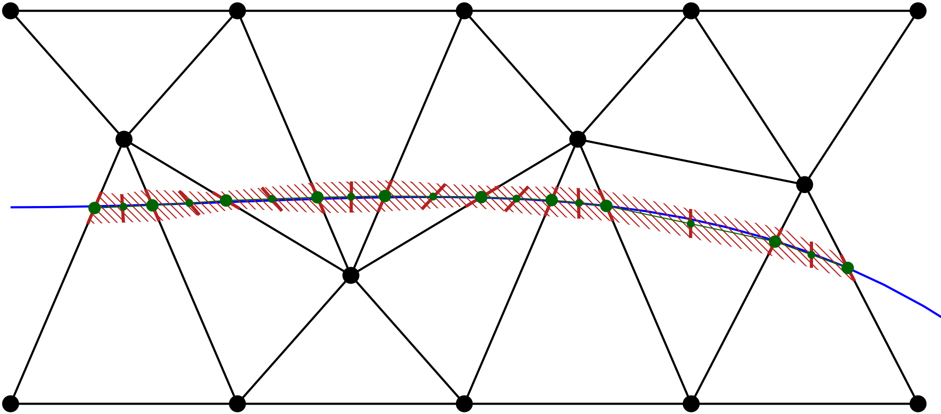
Tubular Neighbourhoods

At the vertices of the triangulation of $\mathcal{M}$ (green), use an interpolation of the normal directions.



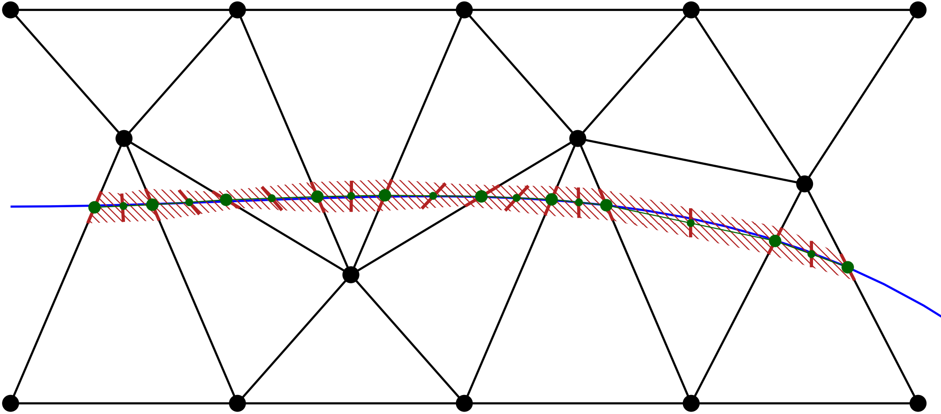
Tubular Neighbourhoods

Linearly extend the ‘normal’ directions to each point of the triangulation. The technically difficult thing to prove is that the normal direction do not change to much so that there is a lower bound on the size of the tubular neighbourhood.



Tubular Neighbourhoods

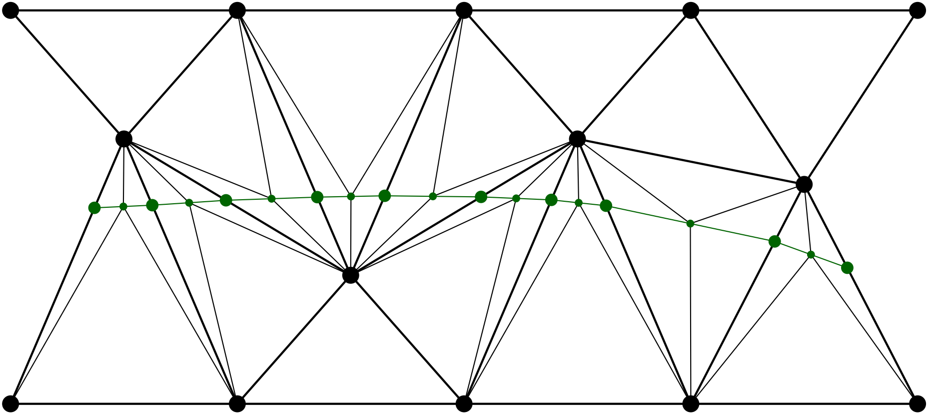
The homeomorphism follows because the ‘normal’ directions intersect the manifold transversally and the manifold and its triangulation are close.



Remark and take home message

Remark

The triangulation is a part of the barycentric subdivision the ambient triangulation with some vertices shifted.



Take home message

- Whitney's method gives an algorithm (which has reasonably good complexity)
- The guarantees are now all quantified in terms of the reach
- There is now a completely elementary proof (in the sense that no topological lemma is used)

Questions?

Interpolation of normal spaces

Consider the orthogonal projection map $\pi_{v(\tau_k^{d-n}),p} : \mathrm{aff}(\tau_k^{d-n}) \to N_p\mathcal{M}$.
For any $w \in N_p\mathcal{M}$, define

$$N_{\tau,p}(w) = \frac{1}{j}\left(\pi^{-1}_{v(\tau_1^{d-n}),p}(w) + \cdots + \pi^{-1}_{v(\tau_j^{d-n}),p}(w)\right).$$

To construct the normal space at $v(\tau)$, pick $p = \pi_{\mathcal{M}}(v(\tau))$ and define $\mathcal{N}_{v(\tau)} = \mathrm{span}(N_{\tau,\pi_{\mathcal{M}}(v(\tau))}(w))$.
Let $\sigma^n = \{v(\tau_0^{d-n}), \ldots, v(\tau_n^d)\} \in K$. For $\bar{p} \in \sigma^n$ with barycentric coordinates $\lambda = (\lambda_0, \ldots, \lambda_n)$, and $w \in N_p\mathcal{M}$ define

$$N_{\bar{p},p}(w) = \lambda_0 N_{\tau_0^{d-n},p}(w) + \cdots + \lambda_n N_{\tau_n^d,p}(w).$$

Set $p = \pi_{\mathcal{M}}(\bar{p})$. By defining $\mathcal{N}_{\bar{p}} = \mathrm{span}(N_{\bar{p},\pi_{\mathcal{M}}(\bar{p})}(w))$, we get affine spaces for each point in each $\sigma^n \in K$.