Delaunay triangulations of a family of symmetric hyperbolic surfaces in practice

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Ph.D. defense
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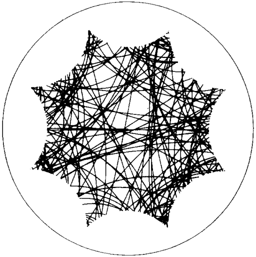
Most symmetric surface of genus 2: Bolza

Mathematical physics

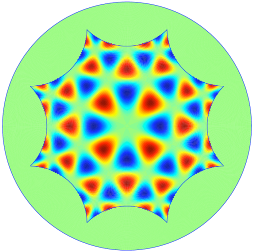


[Balazs, Voros '86]

chaos

[Sausset, Tarjus, Viot '08]

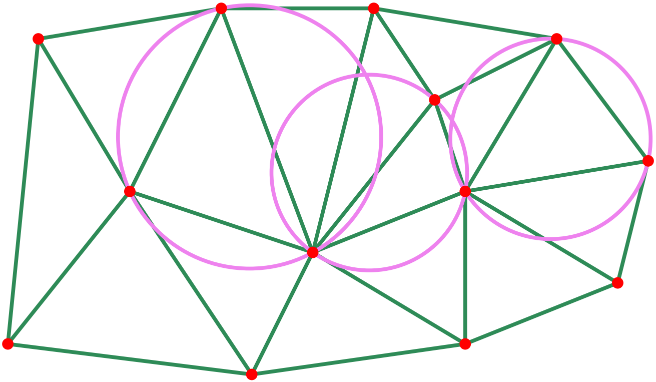
glass forming liquid



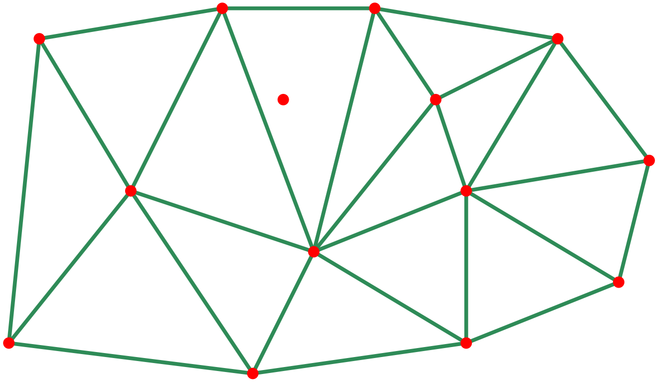
[Chossat, Faye, Faugeras '11]

visual perception of textures

What is a Delaunay triangulation?



Bowyer's algorithm



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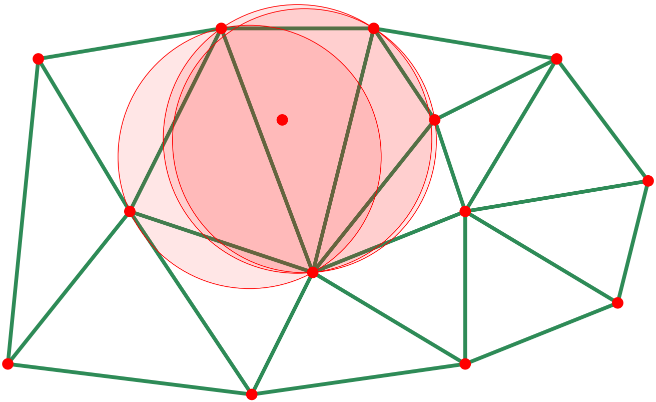

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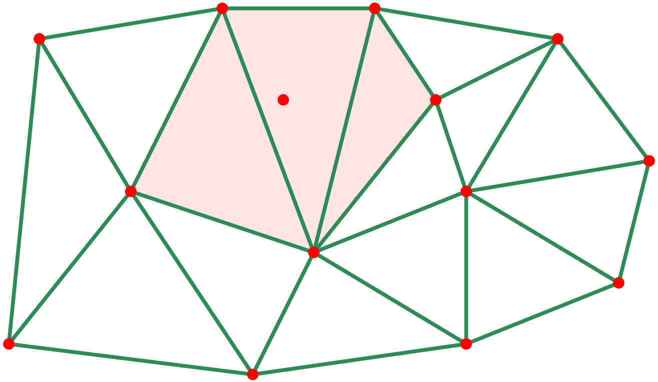

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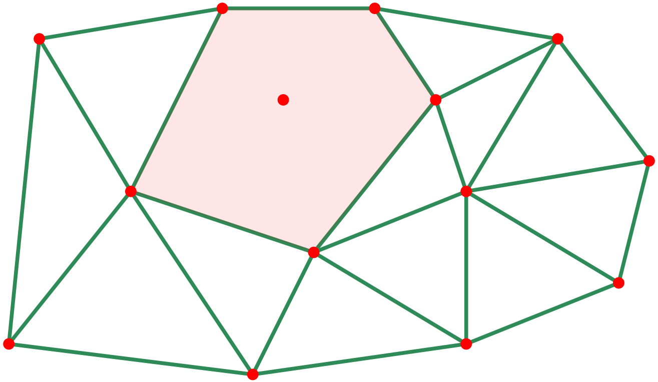

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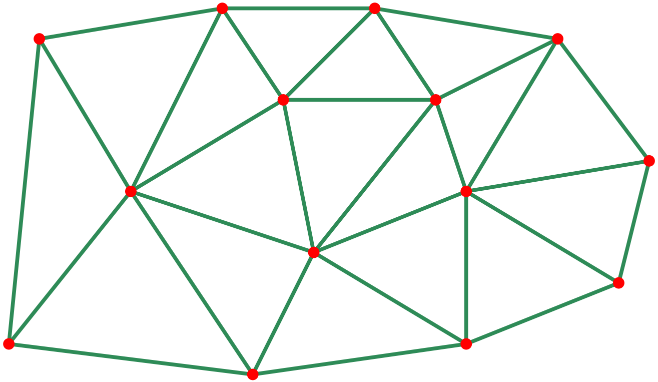


The Computational Geometry Algorithms Library CGAL

- CGAL project: founded in 1996
- CGAL library: open source since 2003
- Today:
  - open-source library for CG
  - over 100 packages (triangulation, mesh, arrangements...)
  - used in GIS, CAD, medical imaging, robotics...

Our goal:

implement triangulations of hyperbolic surfaces in CGAL

State of the art

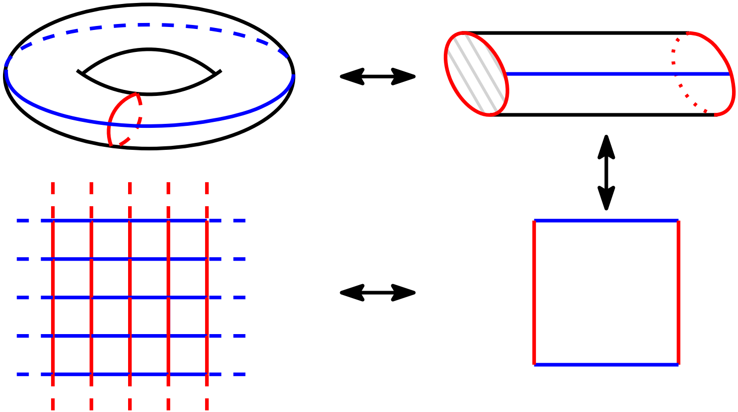
Closed Euclidean manifolds

- Algorithms
  - 2D [Mazón, Recio '97]
  - 3D [Dolbilin, Huson '97]
  - dD [Caroli, Teillaud '16]
- Software (square/cubic flat torus)
  - 2D [Kruithof '13]
  - 3D [Caroli, Teillaud '09]

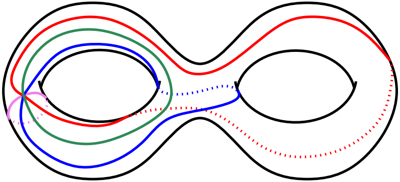
Closed hyperbolic manifolds

- Algorithms
  - 2D, genus 2 [Bogdanov, Teillaud, Vegter, SoCG'16]

Cutting open a flat torus



Cutting open a double torus



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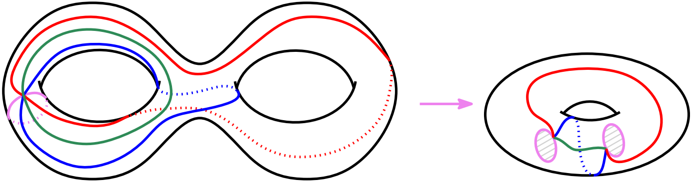

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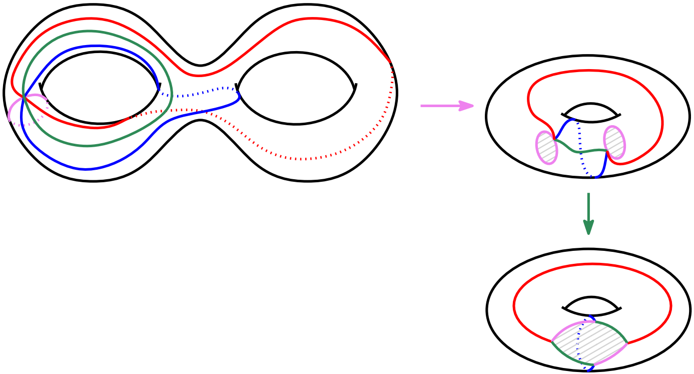

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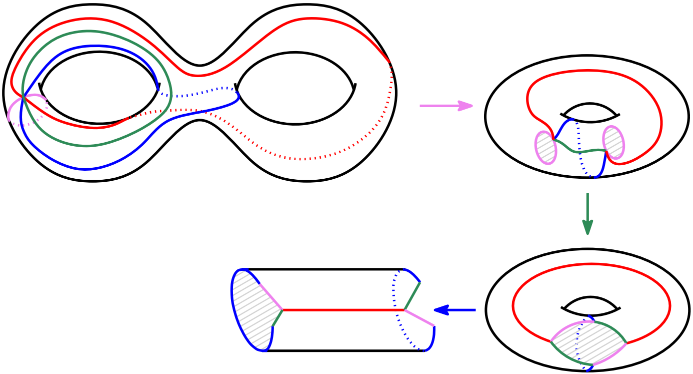

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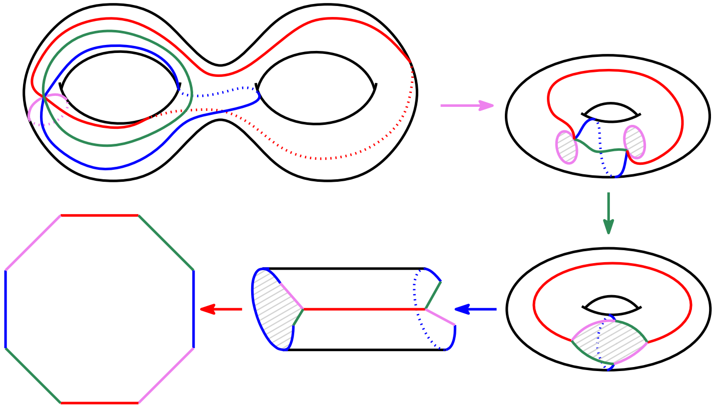
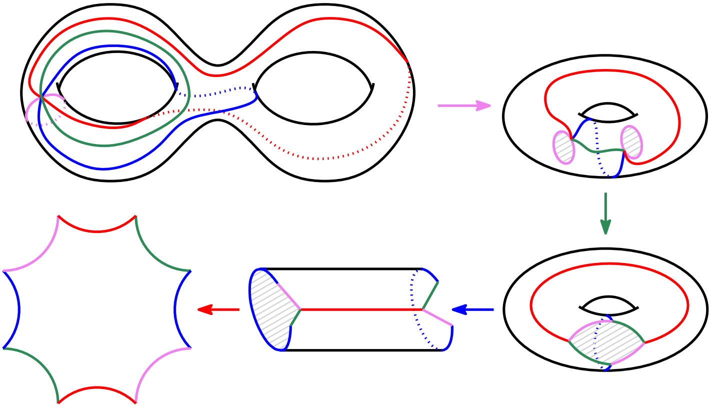


cannot tile Euclidean plane!

Cutting open a double torus



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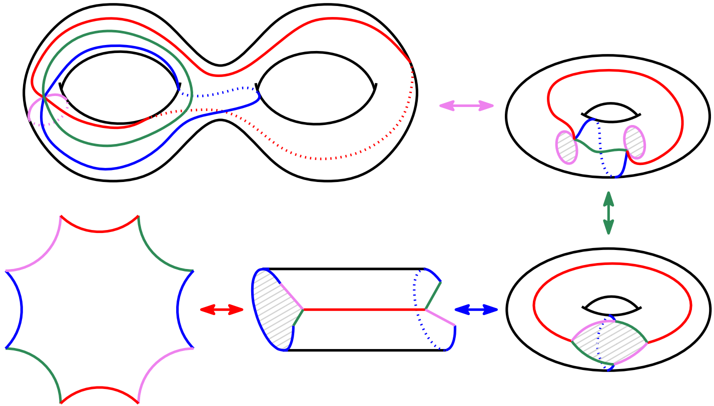
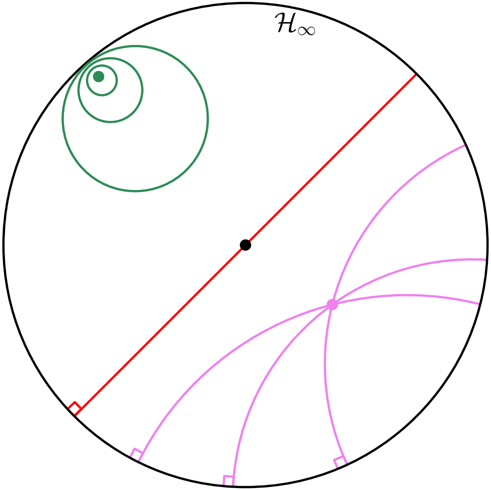
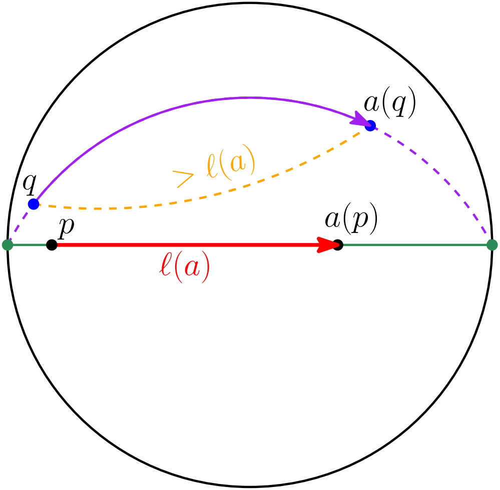


Poincaré model of the hyperbolic plane $\mathbb{H}^2$



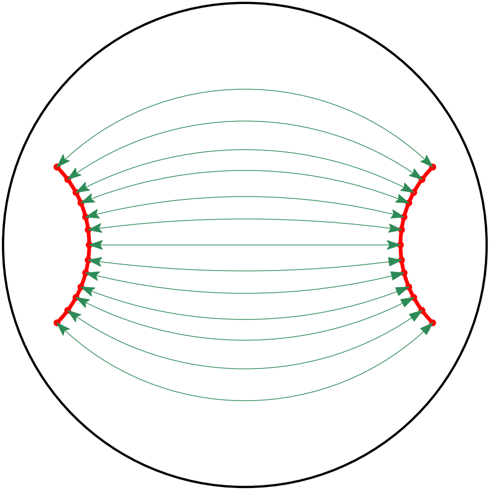
Hyperbolic translations

Special case: axis = diameter



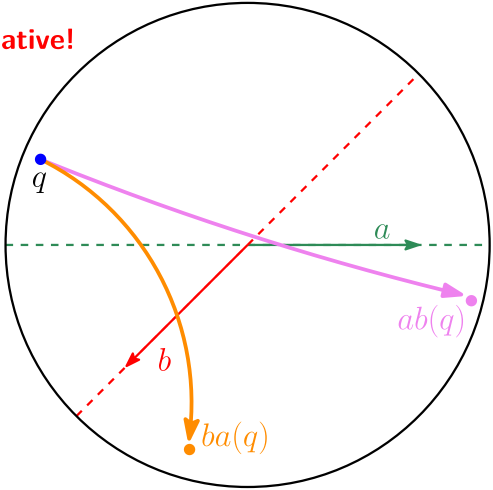
Hyperbolic translations

Side-pairing transformation

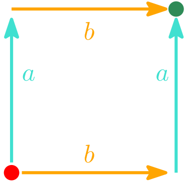


Hyperbolic translations

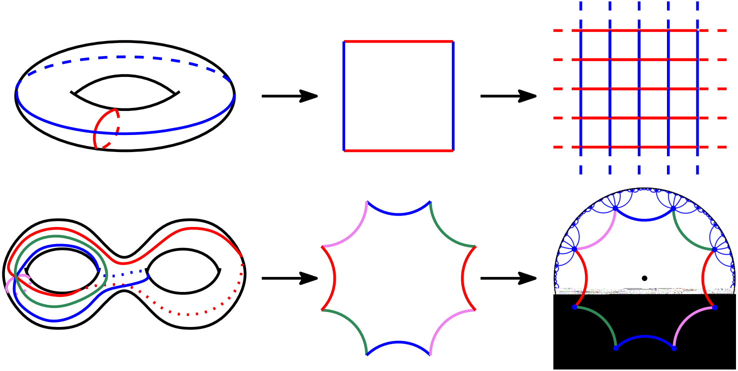
Non-commutative!



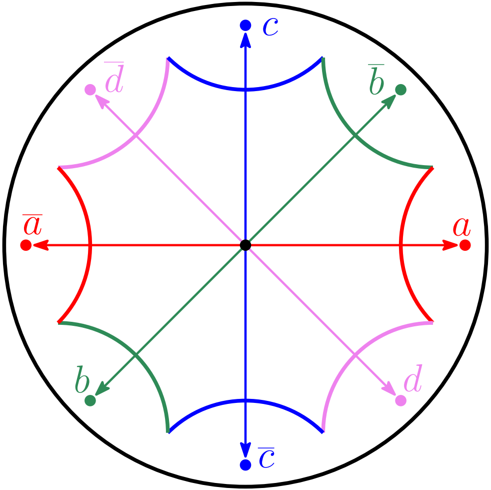
Compare with Euclidean:



Tilings of the Euclidean and hyperbolic planes



Tiling of the hyperbolic plane with octagons



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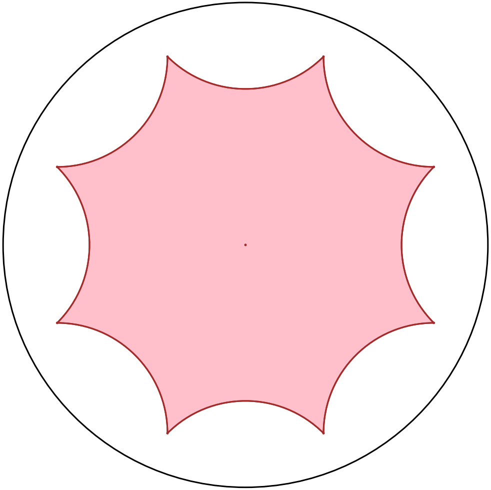

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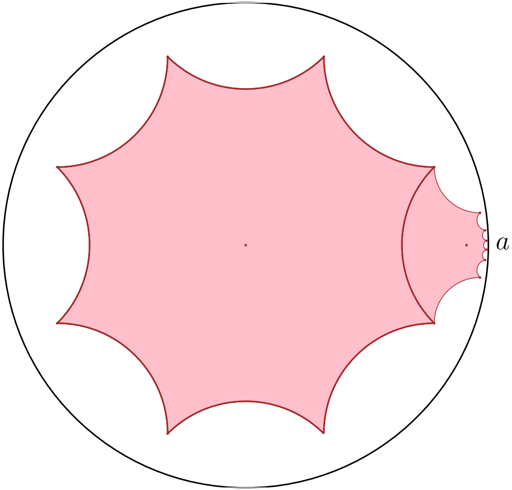

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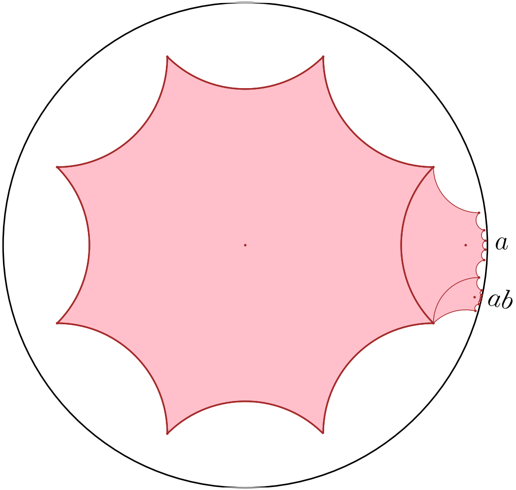

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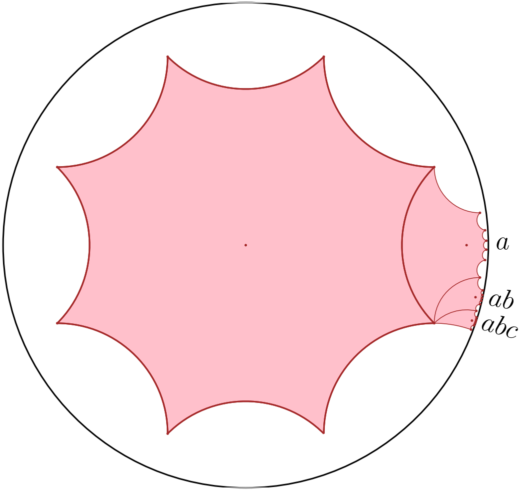

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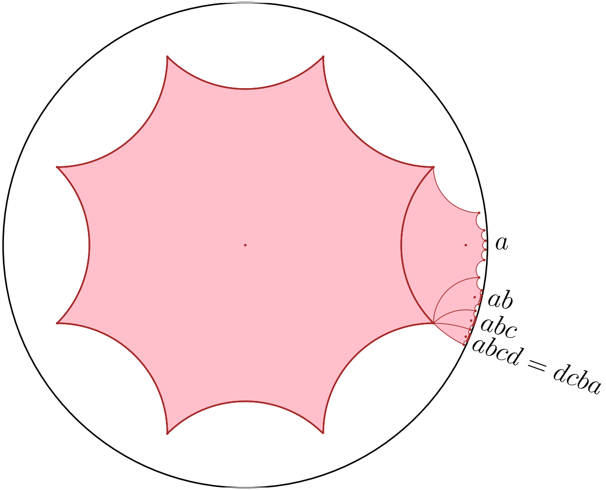

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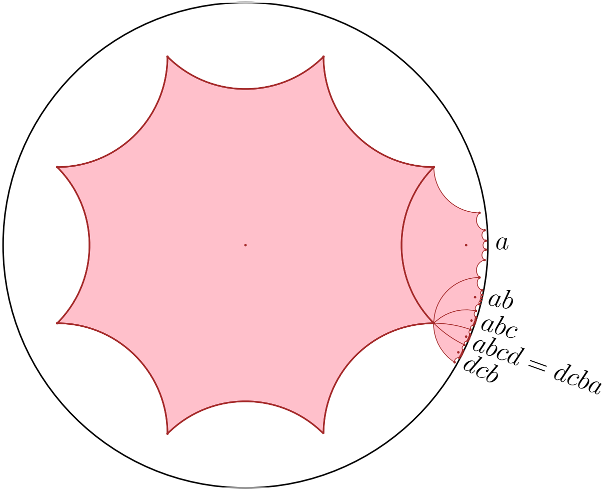

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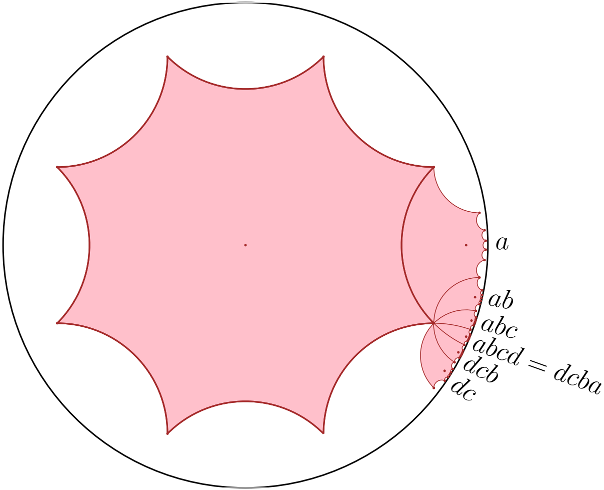

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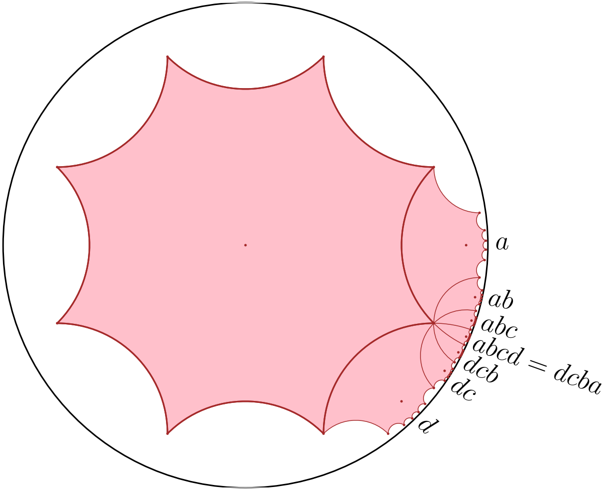

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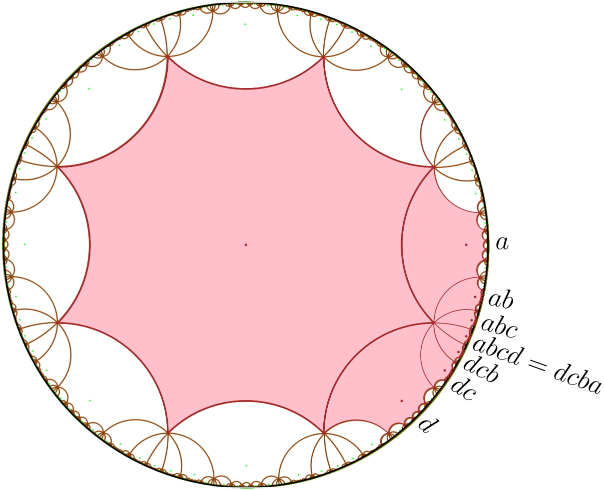
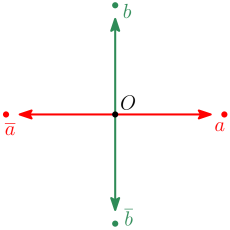


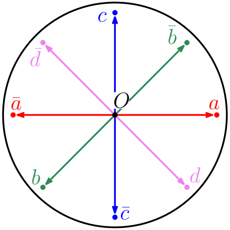
The square flat torus and the Bolza surface



Euclidean: translation group

$$\Gamma_1 = \left\langle a, b \mid ab\bar{a}\bar{b} = \mathbb{1} \right\rangle$$

Flat torus: $\mathbb{M}_1 = \mathbb{E}^2/\Gamma_1$
with projection map $\pi_1 : \mathbb{E}^2 \to \mathbb{M}_1$

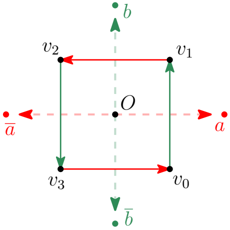


Hyperbolic: Fuchsian group

$$\Gamma_2 = \left\langle a, b, c, d \mid abcd\bar{a}\bar{b}\bar{c}\bar{d} = \mathbb{1} \right\rangle$$

Bolza surface: $\mathbb{M}_2 = \mathbb{H}^2/\Gamma_2$
with projection map $\pi_2 : \mathbb{H}^2 \to \mathbb{M}_2$

The square flat torus and the Bolza surface



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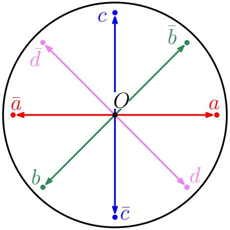

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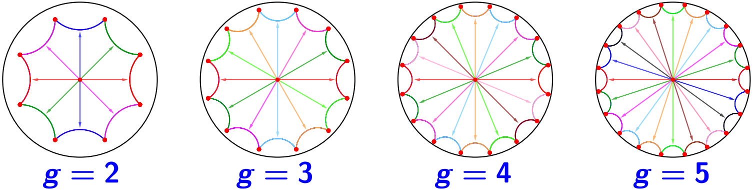
Symmetric hyperbolic surfaces of genus $g \geq 2$



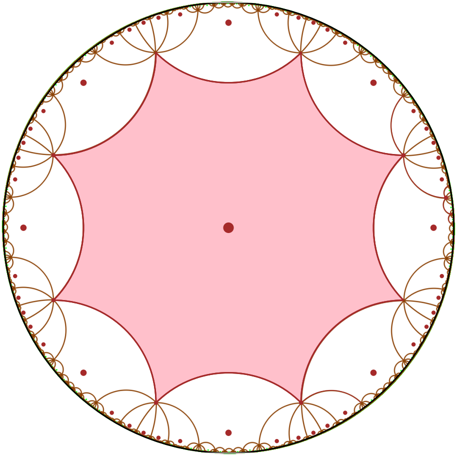
angle sum $= 2\pi$ for all $4g$-gons!

Let Γ_g: Fuchsian group with finite presentation similar to Bolza
$\rightarrow$ $2g$ generators, single relation

Symmetric hyperbolic surface: $\mathbb{M}_g = \mathbb{H}^2 / \Gamma_g$, $\quad g \geq 2$

with natural projection mapping $\pi_g : \mathbb{H}^2 \rightarrow \mathbb{M}_g$

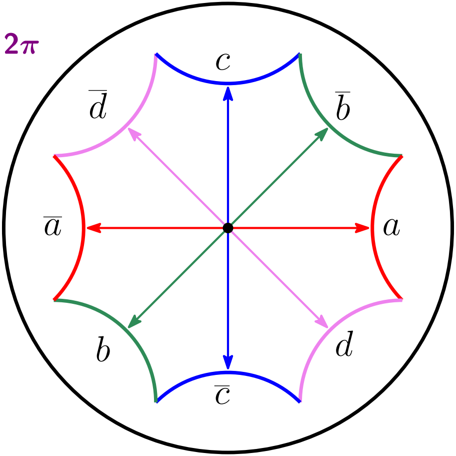
Dirichlet regions



Voronoi diagram of $\Gamma_g O$ for $g = 2$

Dirichlet regions

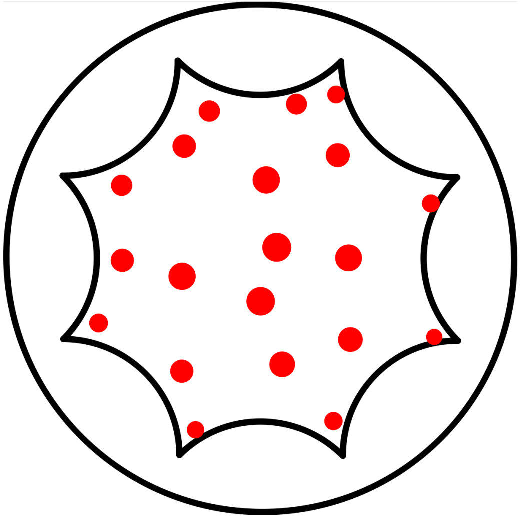
angle sum $= 2\pi$



Fundamental domain D_g = Dirichlet region of O for Γ_g

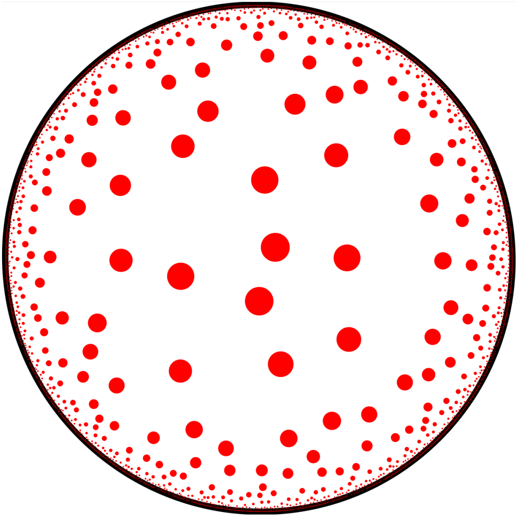
here for $g = 2$

Delaunay triangulation



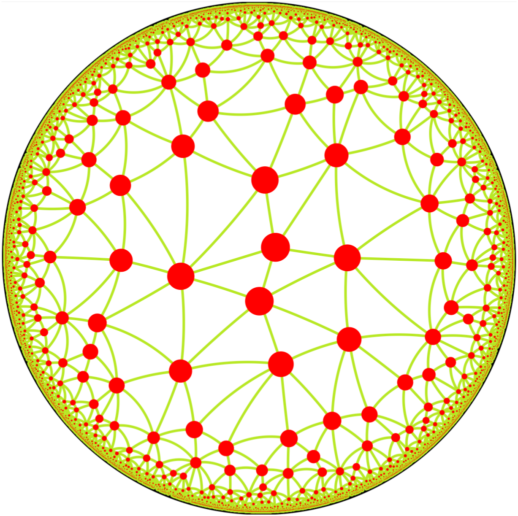
S set of points in D_g

Delaunay triangulation



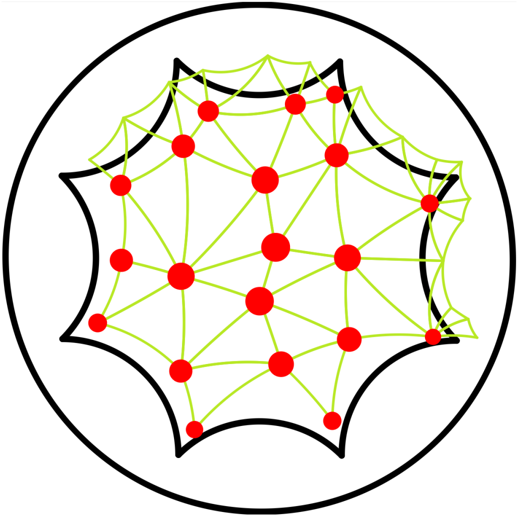
orbits $\Gamma_g S$ in $\mathbb{H}^2$

Delaunay triangulation



Delaunay triangulation in $\mathbb{H}^2$
$DT_{\mathbb{H}}(\Gamma_g S)$

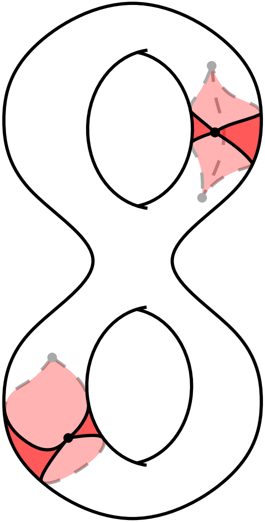
Delaunay triangulation



Delaunay triangulation of $\mathbb{M}_g$

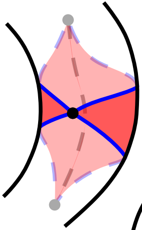
$DT_{\mathbb{M}_g}(S)$

Delaunay triangulation

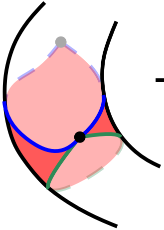


projection of $DT_{\mathbb{H}}(\Gamma_g S)$ on the surface $\mathbb{M}_g$

→ not necessarily a simplicial complex!



→ **double edges**

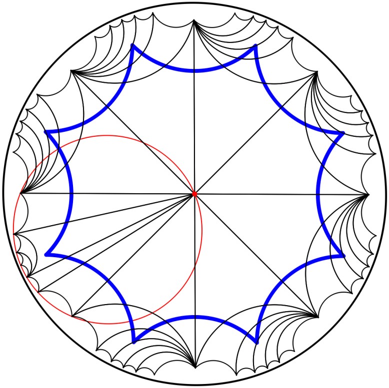


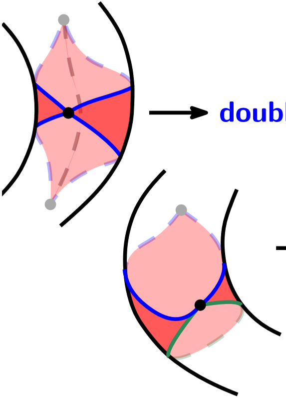
→ **double edges and/or loops**

Delaunay triangulation

projection of $DT_{\mathbb{H}}(\Gamma_g S)$ on the surface $\mathbb{M}_g$

→ not necessarily a simplicial complex!



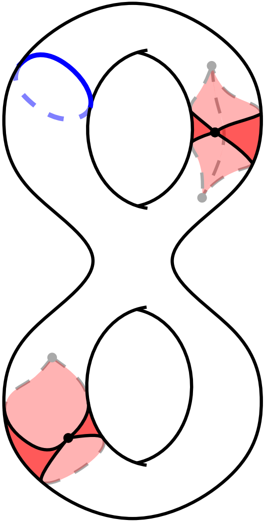


→ **double edges**

→ **double edges and/or loops**

Validity condition

[BTV16]



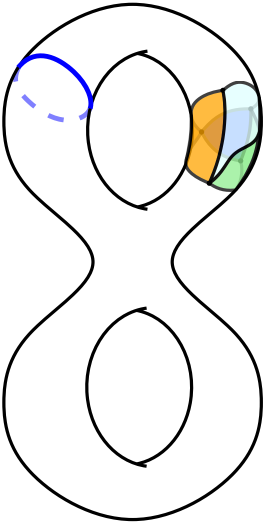
projection of $DT_{\mathbb{H}}(\Gamma_g S)$ on the surface $\mathbb{M}_g$

$\rightarrow$ not necessarily a simplicial complex!

Systole of a surface = minimum length of a non-contractible loop on the surface

Validity condition

[BTV16]



projection of $DT_{\mathbb{H}}(\Gamma_g S)$ on the surface $\mathbb{M}_g$

$\rightarrow$ **is a simplicial complex, if**

$$\delta_S < \frac{1}{2}\text{sys}(\mathbb{M}_g), \quad \text{where}$$

δ_S = diameter of largest disks in $\mathbb{H}^2$ not containing any point of $\Gamma_g S$

$$\boldsymbol{DT_{\mathbb{M}_g}(S) := \pi_g(DT_{\mathbb{H}}(\Gamma_g S))}$$

Computing Delaunay triangulations of $\mathbb{M}_g$

Use set of *dummy points Q_g* that satisfies the validity condition:

$$S := Q_g \bigcup P \Longrightarrow \delta_S < \frac{1}{2}\mathrm{sys}(\mathbb{M}_g) \quad \text{always}$$

Computing Delaunay triangulations of $\mathbb{M}_g$

Use set of *dummy points* Q_g that satisfies the validity condition:

$$S := Q_g \bigcup P \Longrightarrow \delta_S < \frac{1}{2}\text{sys}(\mathbb{M}_g) \quad \text{always}$$

Algorithm for Delaunay triangulations of $\mathbb{M}_g$ [BTV16]

1. initialize $DT_{\mathbb{M}_g}$ with a set of dummy points Q_g
2. insert input points P in the triangulation
3. remove points of Q_g from the triangulation, if possible

→ condition preserved with insertion of new points
→ final triangulation might contain dummy points

Problem statement

To compute $DT_{\mathbb{M}_g}(S)$, we need to *choose* **what** to store.

Requirement: all input points lie in D_g
$\rightarrow$ unique representative in $D_g \subset \mathbb{H}^2$ for each point on $\mathbb{M}_g$

Question: How to choose a unique representative for each *face*?

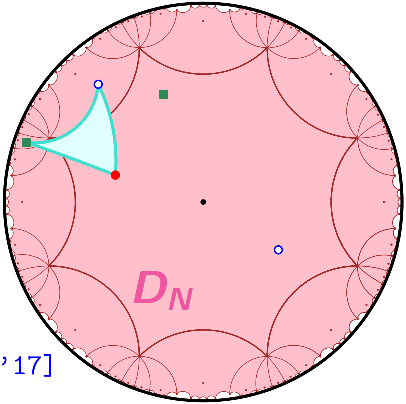
Inclusion property

Let $S \subset D_g$ be a point set such that

$$\delta_S < \frac{1}{2}\,\text{sys}(\mathbb{M}_g).$$

Let σ be a face of $DT_{\mathbb{H}}(\Gamma_g S)$ with at least one vertex in D_g

$\Rightarrow$ σ is contained in D_N

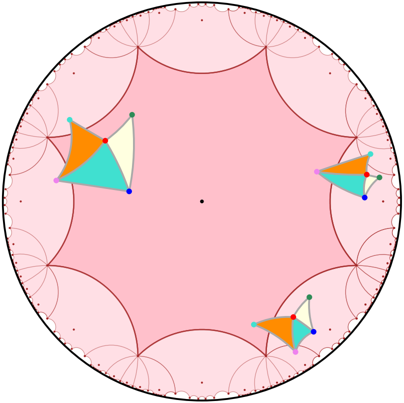


Proof:

- for $g = 2$ $\rightarrow$ [IT, SoCG '17]
- for $g \geq 2$ $\rightarrow$ [EITV]

Canonical representatives of faces

Canonical representative: face with at least one vertex in D_g
$\rightarrow$ other vertices will be in D_N



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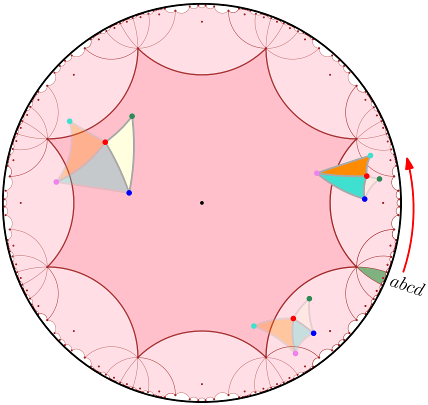
To make it unique:
$\rightarrow$ choose the face "closest" to the first Dirichlet neighbor



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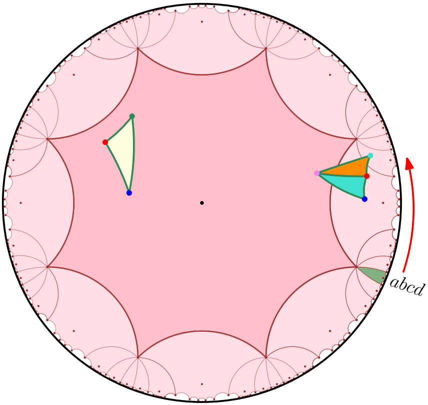
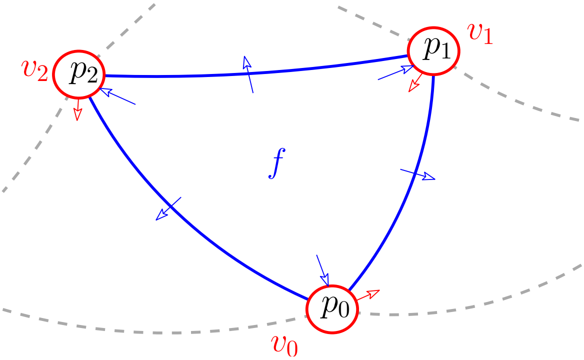
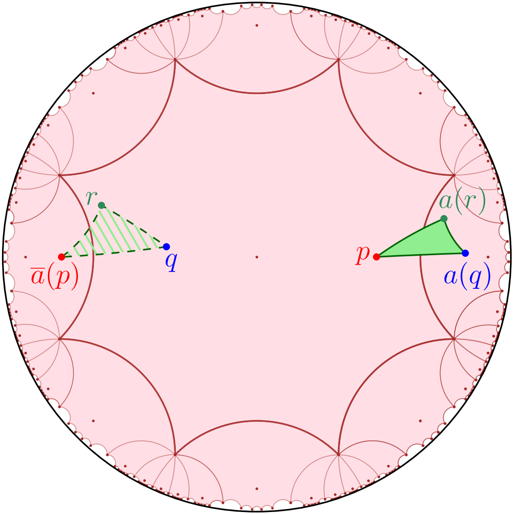


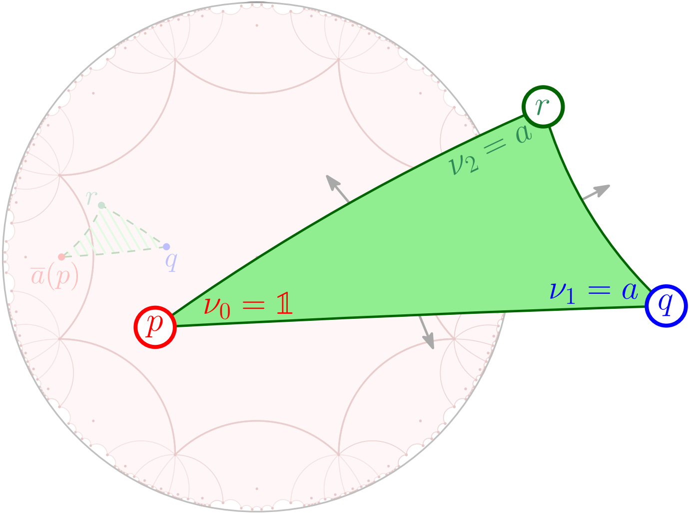
CGAL triangulation data structure



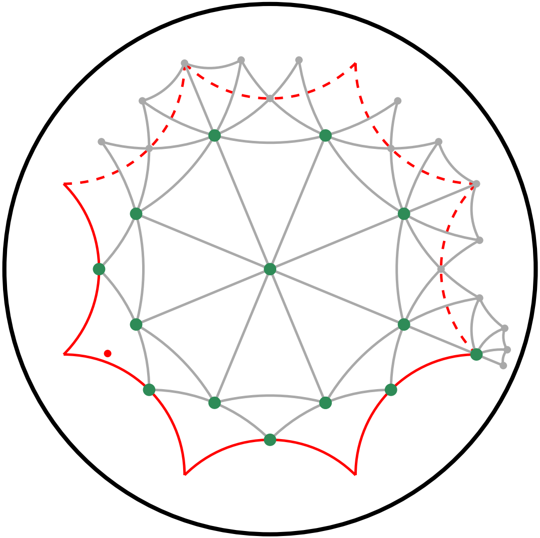
Canonical representatives can cross the boundary



CGAL extended triangulation data structure



Point Location



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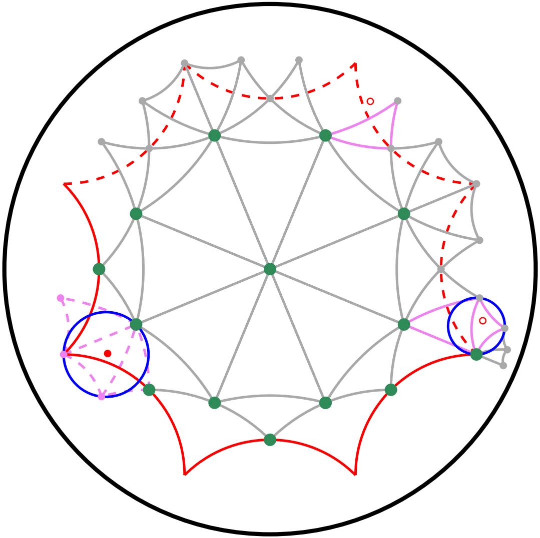
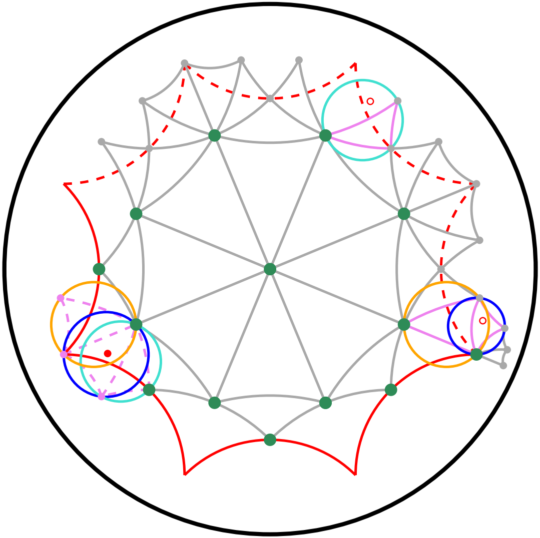


Point Location



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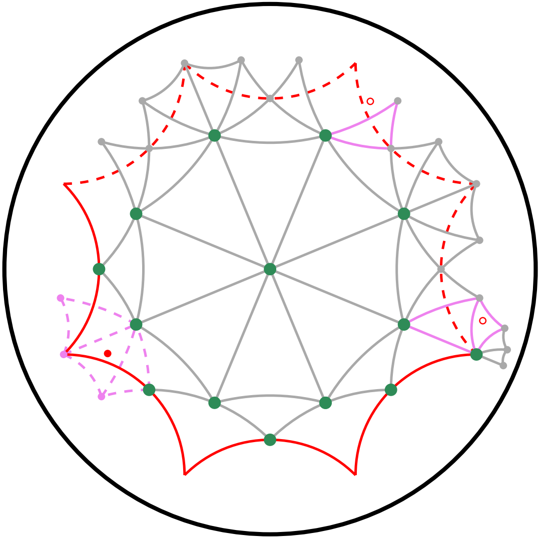
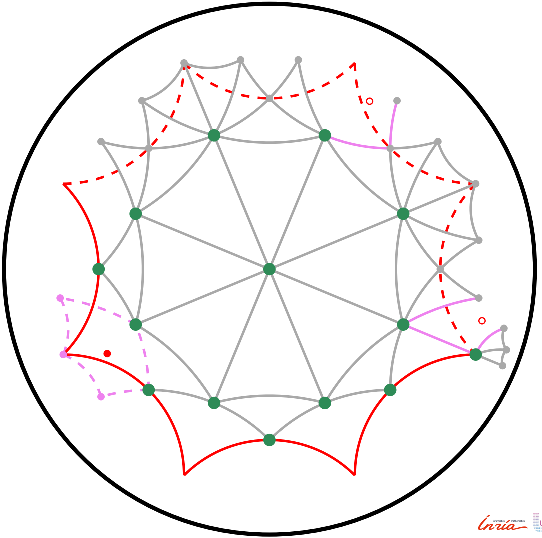


Point Insertion

recall:
"hole" = topological disk



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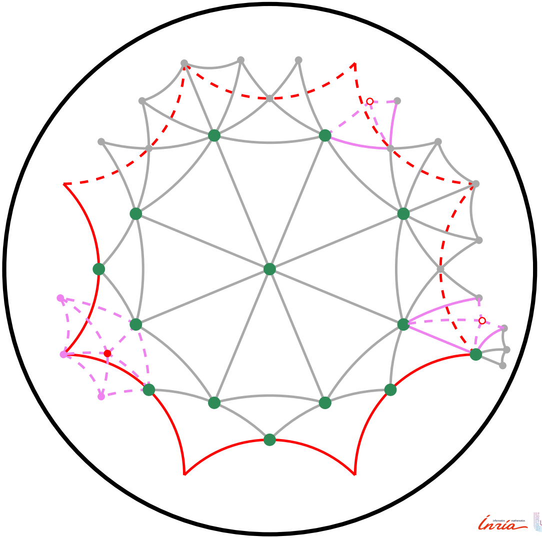
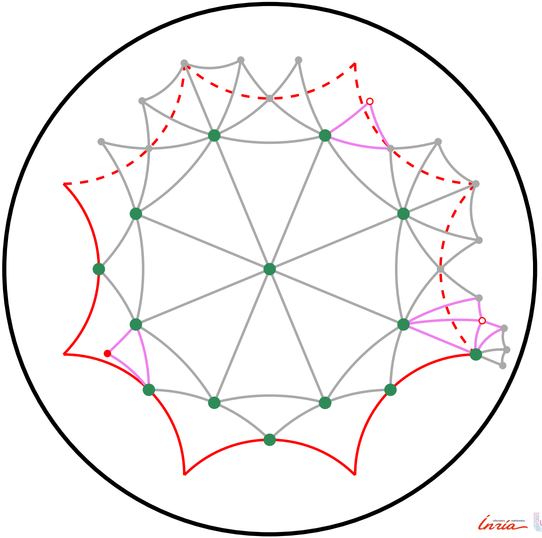


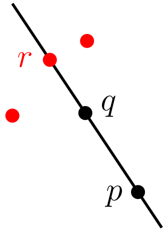
Point Insertion

Computations
on translations

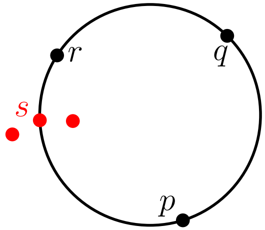


Predicates

$$Orientation(p, q, r) = \text{sign} \begin{vmatrix} p_x & p_y & 1 \\ q_x & q_y & 1 \\ r_x & r_y & 1 \end{vmatrix}$$



$$InCircle(p, q, r, s) = \text{sign} \begin{vmatrix} p_x & p_y & p_x^2 + p_y^2 & 1 \\ q_x & q_y & q_x^2 + q_y^2 & 1 \\ r_x & r_y & r_x^2 + r_y^2 & 1 \\ s_x & s_y & s_x^2 + s_y^2 & 1 \end{vmatrix}$$



Predicates

Suppose that the points in S are rational.

Input of the predicates can be their images under translations, e.g.,

$$\overline{b} : z \mapsto \frac{z \cdot \left(1+\sqrt{2}\right) + e^{\frac{i\pi}{4}}\sqrt{2}\sqrt{1+\sqrt{2}}}{z \cdot e^{-\frac{i\pi}{4}}\sqrt{2}\sqrt{1+\sqrt{2}} + \left(1+\sqrt{2}\right)}.$$

Orientation:

Degree	16	20
# cases	28	42

InCircle:

Degree	32	40	48	56	64	72
# cases	13	57	140	21	6	1

Point coordinates represented with `CORE::Expr`
→ (filtered) exact evaluation of predicates

Experiments

Fully dynamic implementation

1 million rational random points

- CGAL Euclidean DT (double) $\sim$ 1 sec.
- CGAL Euclidean DT (CORE::Expr) $\sim$ 22 sec.
- Hyperbolic periodic DT (double) $\sim$ 13 sec.
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Predicates

- 0.76% calls to predicates involve non-identity translations
- responsible for 36% of total time spent in predicates

No dummy points left after insertion of > 200 random points.

Demo

- Implementation (open source) is available on Github:
  `https://imiordanov.github.io/code/`
  To appear in `CGAL v.4.14` (March 2019)
- YouTube video of CGAL demo shows hyperbolic free motion:
  `https://tinyurl.com/bolza-free-motion`
- We will see the live demo right now!

An initial set of dummy points [EITV]

For $\mathbb{M}_2$, a set of dummy points was given [BTV16]. **In general?**

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1. Start with a set W_g for $\mathbb{M}_g$ (called *Weierstrass points*)
   $\rightarrow$ origin, one vertex, and midpoints of sides of the $4g$-gon
2. Compute the images of these points in D_N
3. Compute their hyperbolic Delaunay triangulation in $\mathbb{H}^2$
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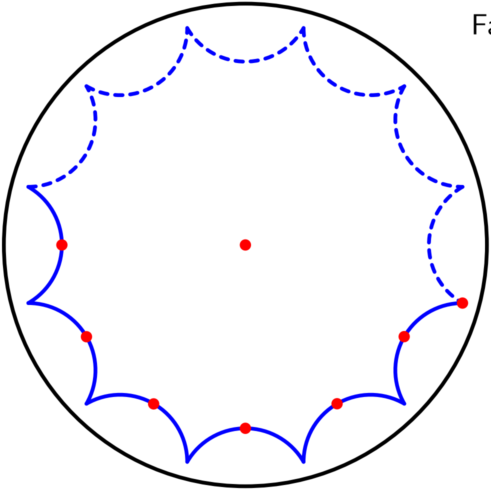
- $\mathrm{sys}(\mathbb{M}_g) = 2\,\mathrm{arcosh}\left(1 + 2\cos\left(\frac{\pi}{2g}\right)\right)$ [Ebbens, 2018]
- triangulation of sets including W_g: contained in D_N

Triangulations of Weierstrass points



Faces with a vertex in the polygon
$\rightarrow$ contained in $\boldsymbol{D_N}$

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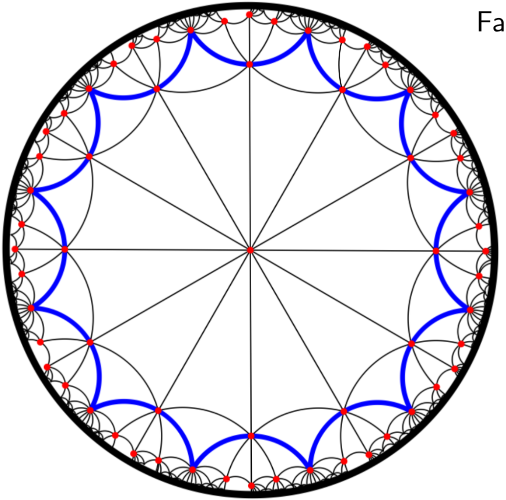
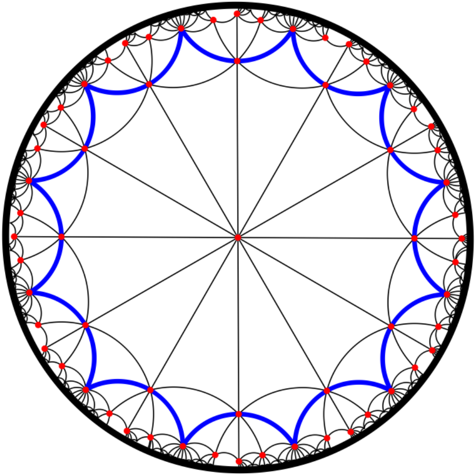


Faces with a vertex in the polygon
$\rightarrow$ contained in $\boldsymbol{D_N}$

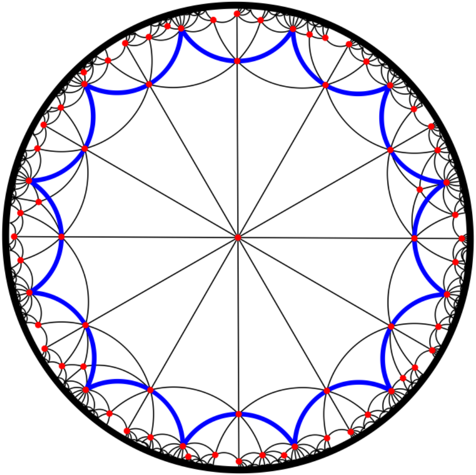
Compute dummy points:

1. Get triangulation in $\boldsymbol{D_N}$
2. Refine triangulation
3. Take points in $4g$-gon

Sequential strategy



Sequential strategy



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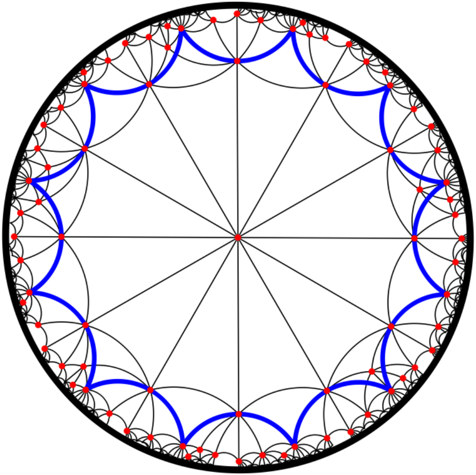
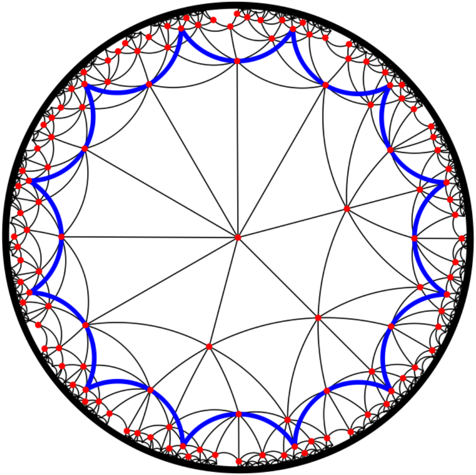
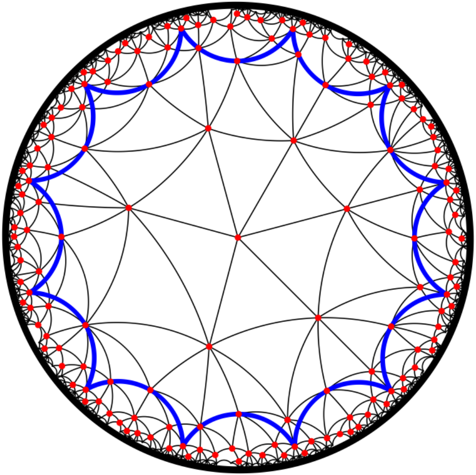


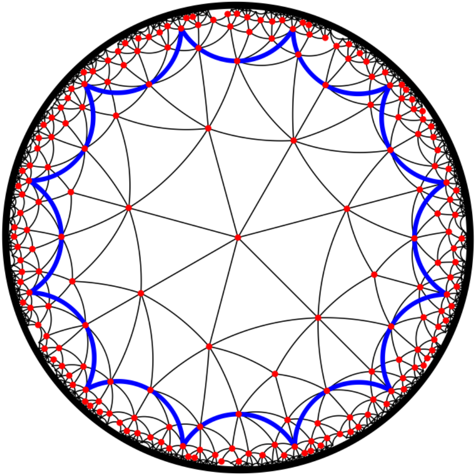
Sequential strategy



Sequential strategy

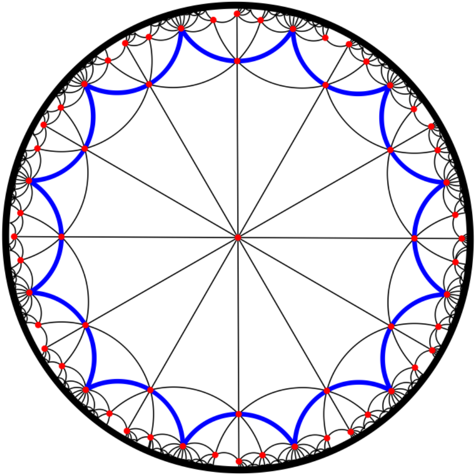


Sequential strategy

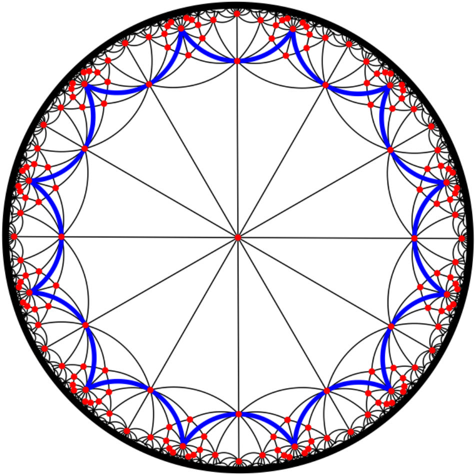


set size $\mathcal{O}(g)$
[EITV]

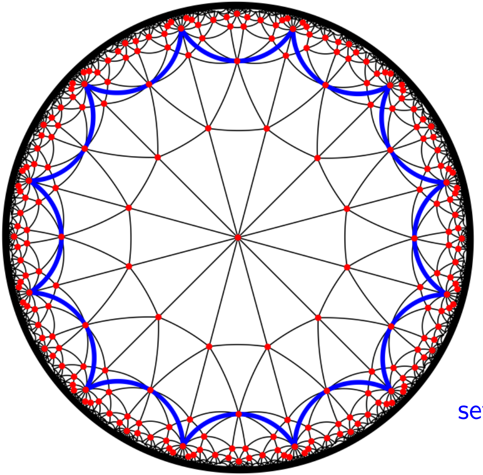
Sequential strategy with symmetries



Sequential strategy with symmetries



Sequential strategy with symmetries



set size $\mathcal{O}(g \log g)$
[EITV]

Implementation

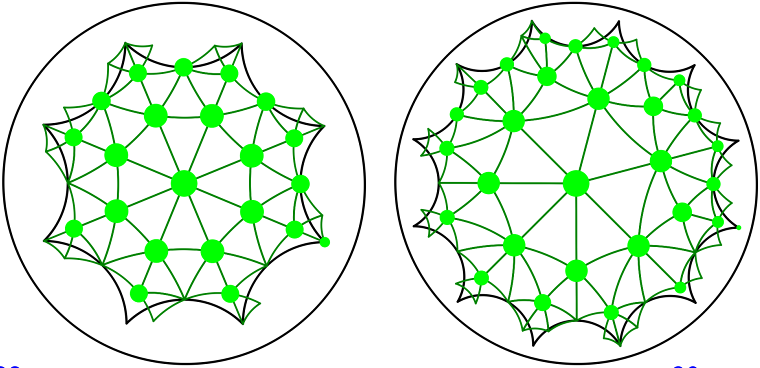
- Preliminary code on Github, but **not** public
- What is implemented:
  - generation of dummy points (first two strategies)
  - initialization of periodic triangulation
  - location, insertion, removal: as for Bolza

Implementation

- Preliminary code on Github, but **not** public
- What is implemented:
  - generation of dummy points (first two strategies)
  - initialization of periodic triangulation
  - location, insertion, removal: as for Bolza
- Problems in practice
  - Recall: exact predicates; now with more complex expressions!
  - Comparison of two numbers: non-conclusive! (even for $g = 3$)
  - Idea: use limited accuracy, validate *a posteriori*
    - obtained preliminary results for $g = 3$
    - up to $2048 \times g$ bits: `CORE` crashes for $g > 3$ (generating Q_g)

Experimental results

sequential

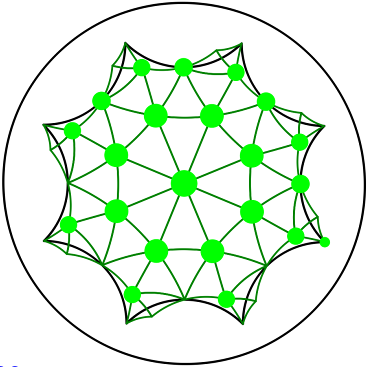


22 pts.

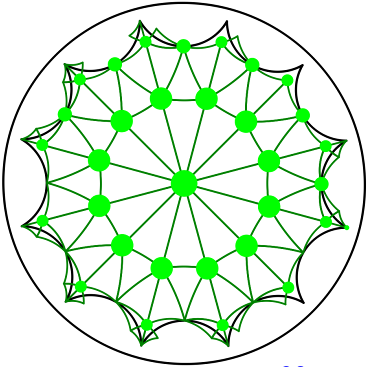
30 pts.

Experimental results

sequential w/ symmetries

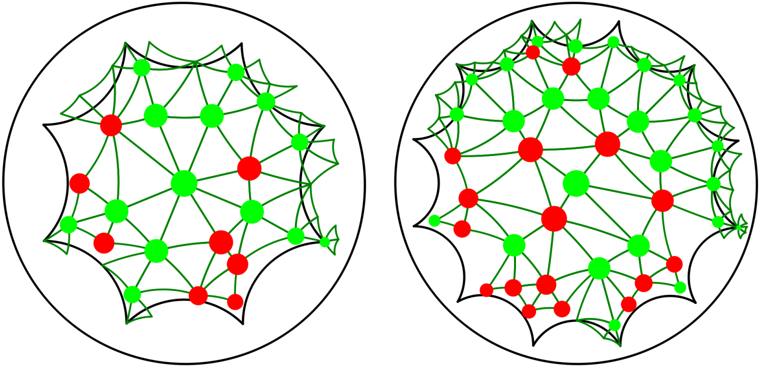


22 pts.

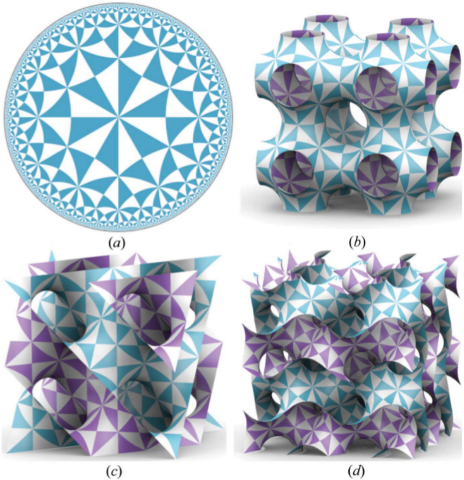


32 pts.

Experimental results insertion/removal



Triply Periodic Minimal Surfaces (TPMS)



[Evans *et al.*, 2013]

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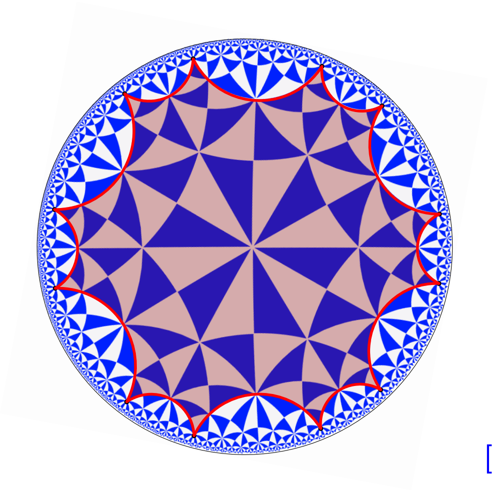
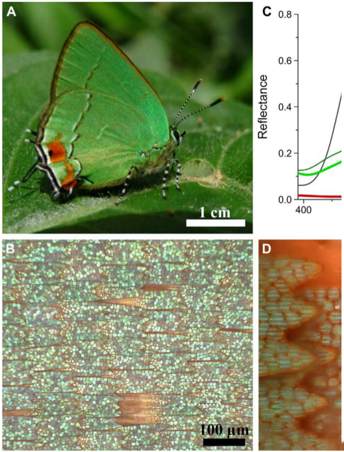
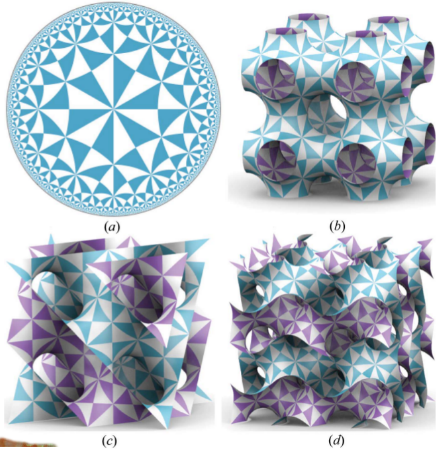


[Evans *et al.*, 2013]

Example of hyperbolic surface: gyroid

[Evans *et al.*, 2013]





[Schröder-Turk *et al.*, 2017]