Computing hyperbolic structures from Thurston's equations

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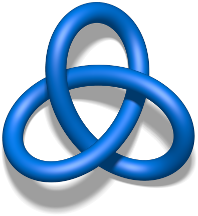
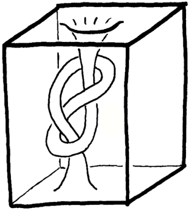
Inria Sophia Antipolis

1er avril 2019

Overview of the talk

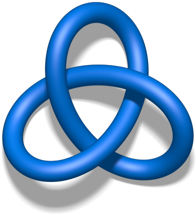
Study knots via their complements.

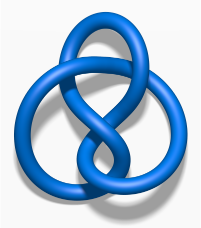




Follows "From angled triangulations to hyperbolic structures" by D. Futer and F. Guéritaud (2012).

Motivation : knot theory





Knots

A knot K is an embedding of S^1 in $\mathbb{R}^3$ via a piecewise linear homeomorphism.

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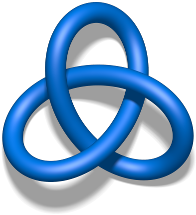

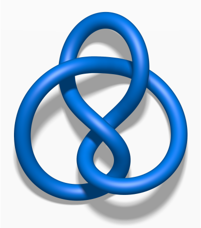

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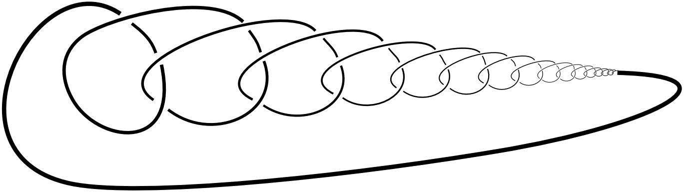
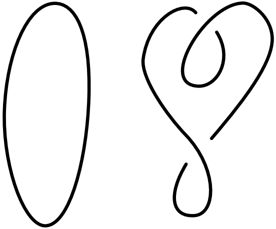


Motivation : knot theory

Ambient isotopy

An ambient isotopy is a continuous deformation of the space.

Two knots are said equivalent if they can be related by an ambient isotopy.



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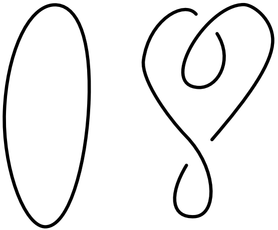

knot invariant

An invariant is a property of the knots which is invariant by ambient isotopy.

3-Manifolds and knots

Gordon-Luecke theorem

If the complements of two picewise linear knots are homeomorphic, then the knots are equivalent.

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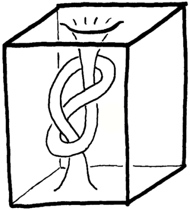

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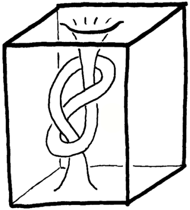

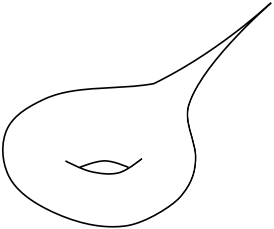

$\rightarrow$ We work with $S^3 \setminus K$. It is an open, orientable, cusped 3-manifold.

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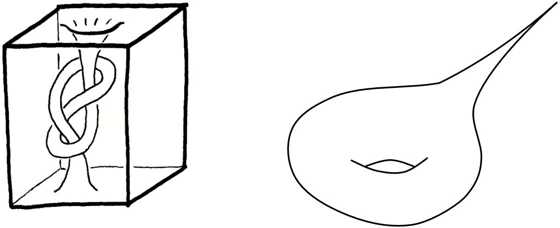
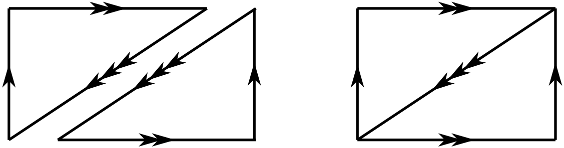


→ We work with $S^3 \setminus K$. It is an open, orientable, cusped 3-manifold.

Hyperbolization theorem (Thurston)

The majority of knot complements admit a hyperbolic structure.

Generalized triangulations of 3-Manifolds



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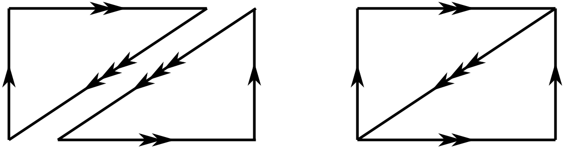

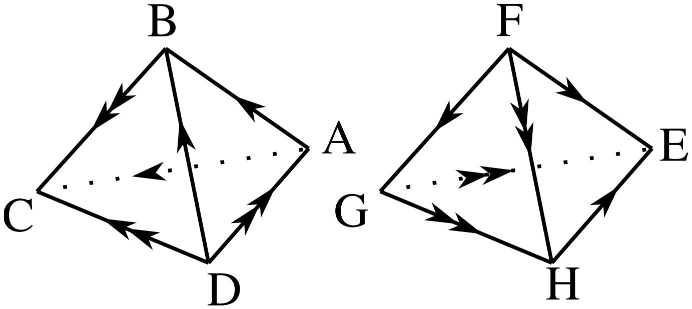

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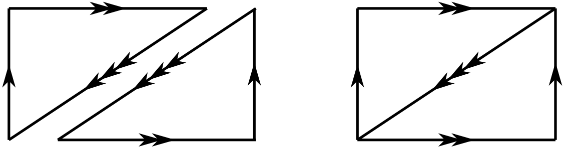
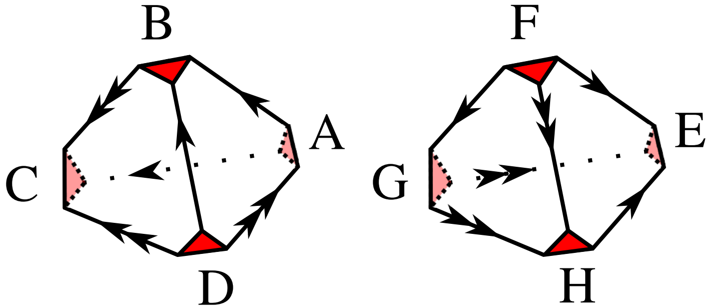
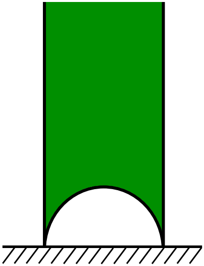




Hyperbolic ideal tetrahedra



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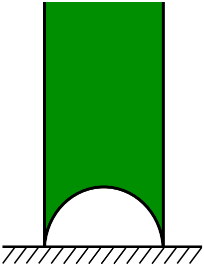
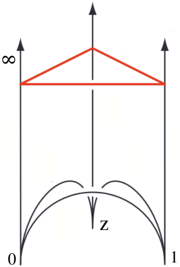
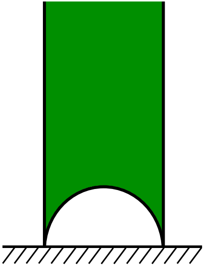


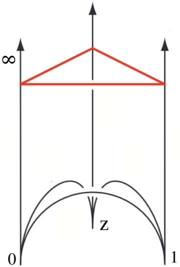


Definition

A hyperbolic ideal tetrahedron is the convex hull of four distinct points on $\partial\mathbb{H}^3$.

Hyperbolic ideal tetrahedra





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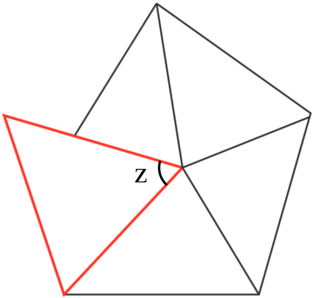
It admits three shape parameters :

$$z, \frac{z-1}{z}, \frac{1}{1-z}$$

Angle structures

Aim

Find the value of the Shape parameters to obtain a hyperbolic structure.



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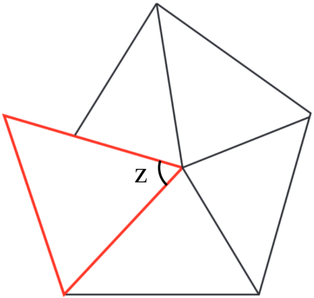
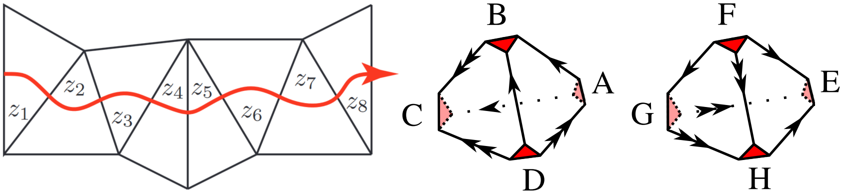


$$\sum_i \log(z_i) = 2i\pi, \quad \forall i, \ Im(z_i) \geq 0$$

Normal curve and holonomy

Normal curve

A sequence of segments cutting the triangles only by their edges.



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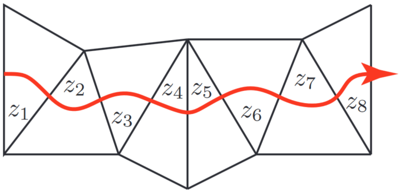

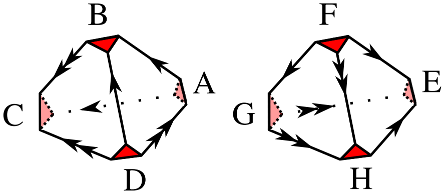

Holonomy

$$H(\sigma) = \sum_i \epsilon_i \log(z_i), \ \ \epsilon_i \in \{-1; +1\}$$

Thurston gluing equations

Edge equations

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Complete hyperbolic structure problem

- Input : triangulation τ of a knot complement.
- Output : complete hyperbolic structure on τ.

SnapPea

- Library by Jeff Weeks.
- Reference tool for the study of 3 manifolds.

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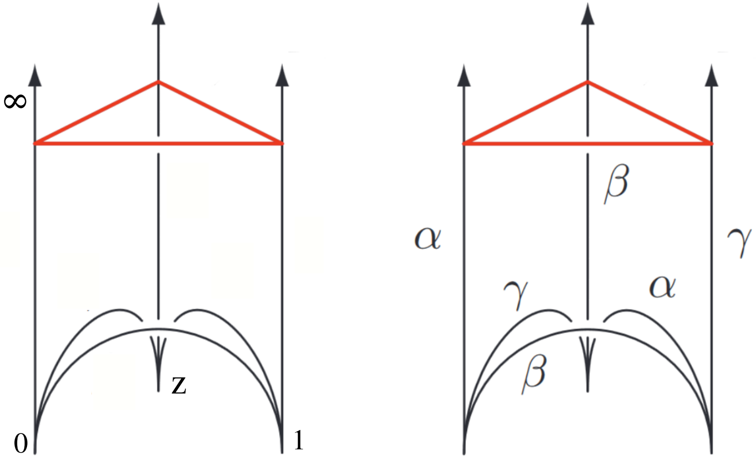
Uses Newton method to directly solve Thurston's equations.

Drawback

Few guaranties on the convergence speed.

The polytope of angle structure



The polytope of angle structure

We drop the complex numbers for angles :

$$z = \frac{sin\ \gamma}{sin\ \beta} \exp^{i\alpha}.$$

- all angles are in $]0,\pi[$;
- the diahedral angles of the tetrahedra sum to π ;
- around each edge, the angles sum to 2π.

Angle structures can be represented in $\mathbb{R}^{3|\tau|}$

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Lemma (Neumann)

With τ the triangulation and $\mathcal{A}(\tau)$ the polytope of angle structures :

$$dim\ \mathcal{A}(\tau) = |\tau| + |\partial M|$$

Existence of an hyperbolic metric

Let :

- M be an orientable 3–manifold with boundary consisting of tori ;
- τ be a corresponding ideal triangulation ;
- $\mathcal{A}(\tau)$ be the polytope of angle structures ;
- $\mathcal{V}$ be the volume functional of $\mathcal{A}(\tau)$.

Theorem (Casson)

If $\mathcal{A}(\tau) \neq \emptyset$, then M admits a complete hyperbolic metric.

Theorem (Casson Rivin)

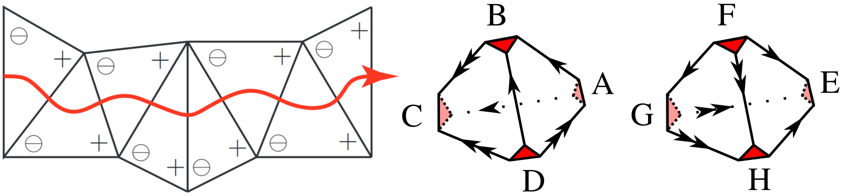
A point $p \in \mathcal{A}(\tau)$ corresponds to a complete hyperbolic metric on the interior of M if and only if p is a critical point of the functional $\mathcal{V} : \mathcal{A}(\tau) \to \mathbf{R}$.

Leading-trailing deformations

Definitions

For a segment crossing a triangle, we denote the leading by $+$ and the trailing corner by $\ominus$ and define them as shown.

Given a curve σ, the associated leading-trailing vector $w(\sigma)$ is the sum of the leading angles minus the sum of the trailing angles of the curve.



Leading trailing deformations

Lemma (Futer Guéritaud)

Let $p \in \mathcal{A}(\tau)$ be an angle structure, and let σ be an oriented normal curve on a cusp of M. Then $w(\sigma)$ is tangent to $\mathcal{A}(\tau)$.

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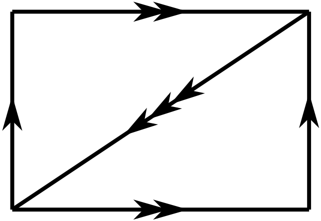
Lemma (Futer Guéritaud)

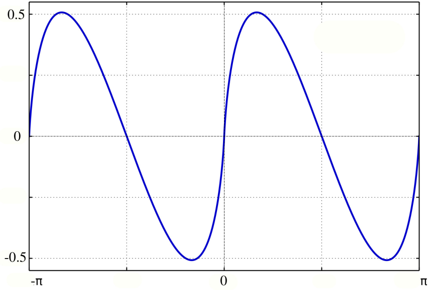
Let $(\sigma_i)_i$ be a family of curves spanning $H_1(\partial M)$, then $(w(\sigma_i))_i$ spans the tangent space of $\mathcal{A}(\tau)$.



Volume of an angle structure

Lobachevsky function

$$Л(x) = -\int_0^x \log|2\sin t|\,\mathrm{d}t$$



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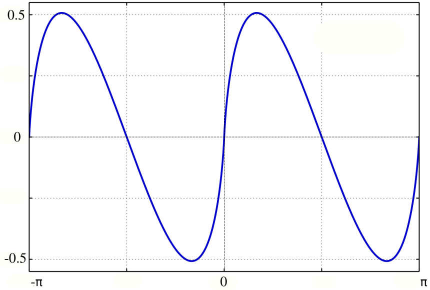

Volume of an ideal tetrahedron

$$\mathcal{V}(\alpha,\beta,\gamma) = Л(\alpha) + Л(\beta) + Л(\gamma)$$

From equation solving to volume maximization

Theorem (Casson Rivin)

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Lemma

$\mathcal{V}$ is strictly concave on $\mathcal{A}(\tau)$.

$\to$ Complete hyperbolic structure problem can be solved via convex optimization.

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Lemma

Let σ be a normal oriented closed curve,

$$\frac{\partial \mathcal{V}}{\partial w(\sigma)} = Re(H(\sigma))$$

Back to equations via Lagrange multipliers

A way of turning constrained optimization problems into equations.

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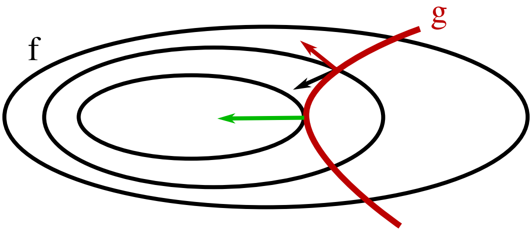
Lagrange conditions

At the optimum $\tilde{x}$ of f under $\vec{g}(x) = \vec{0}$, there exists $\vec{\lambda}$ such that

$$\mathrm{d}(\ f(\tilde{x}) - \vec{\lambda} \cdot \vec{g}(\tilde{x})\ ) = 0$$



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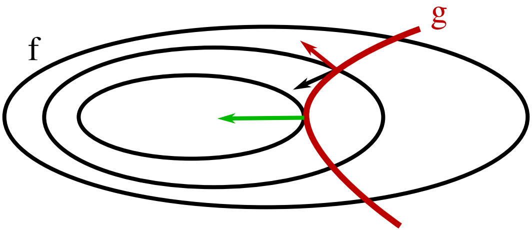

$\rightarrow$ Gives a system of equations with products of sinus alongside the polytope of angle structures.

Conclusion

The presentation

- About finding complete hyperbolic structures on knot complements.
- Follows "From angled triangulations to hyperbolic structures" by D. Futer and F. Guéritaud (2012).

Thurston's gluing equations

- Edge gluing equations.
- Trivial holonomy on boundary tori.
- Solved by SnapPea.

Casson-Rivin

- Existence of complete hyperbolic metric.
- Equivalence with critical point of the volume functional.