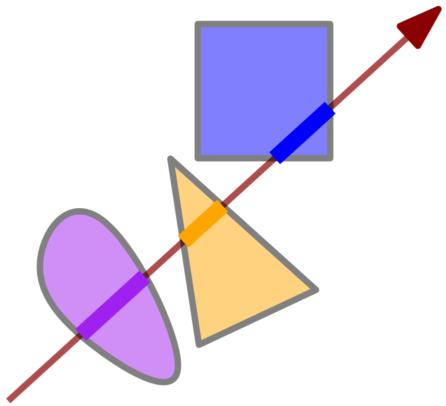
Forbidden Patterns in Geometric Permutations by Combinatorial Lifting

Xavier Goaoc, Andreas Holmsen, Cyril Nicaud

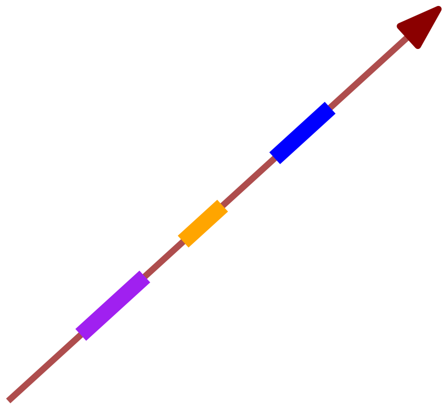
n disjoint compact convex sets in $\mathbb{R}^d$

oriented line that intersects every set
$\rightarrow$ **permutation** (or ordering) of the sets

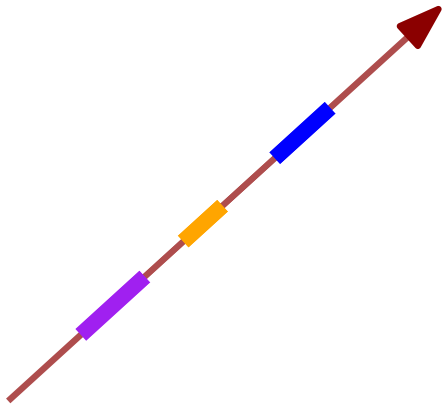


n disjoint compact convex sets in $\mathbb{R}^d$

oriented line that intersects every set
$\rightarrow$ **permutation** (or ordering) of the sets

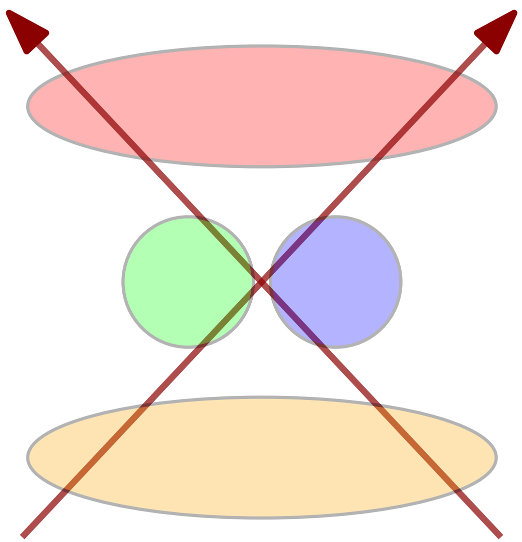


n disjoint compact convex sets in $\mathbb{R}^d$

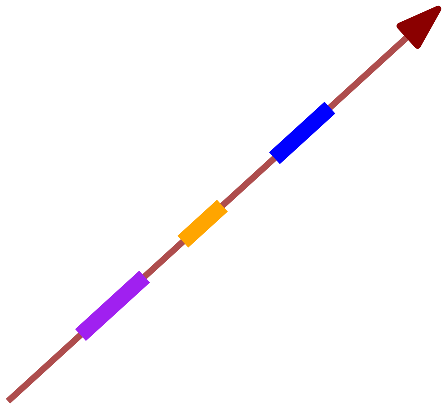


oriented line that intersects every set
$\rightarrow$ **permutation** (or ordering) of the sets

What is the **maximum number** $g_d(n)$ of permutations realizable on n sets in $\mathbb{R}^d$?

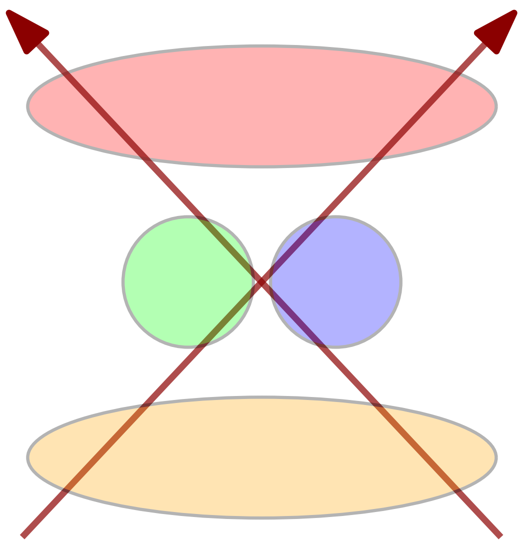


n disjoint compact convex sets in $\mathbb{R}^d$



oriented line that intersects every set
$\rightarrow$ **permutation** (or ordering) of the sets

What is the **maximum number** $g_d(n)$ of permutations realizable on n sets in $\mathbb{R}^d$?



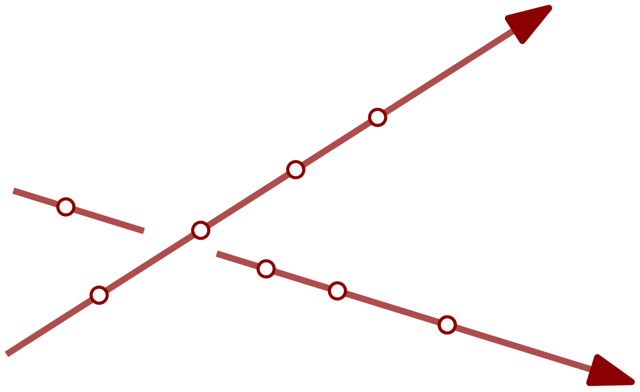
$g_2(n) = 2n - 2$ [1992]

$g_d(n) = O\left(n^{2d-2}\right)$ [1992]

$g_d(n) = \Omega\left(n^{d-1}\right)$ [2002]

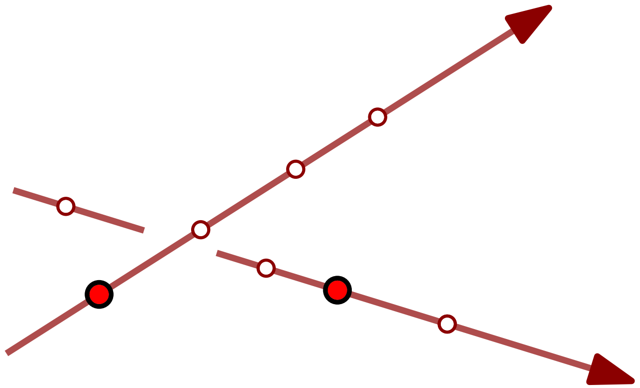
$g_d(n) = O\left(n^{2d-3}\log n\right)$ [2010]

special cases are understood: balls, fatness, ...



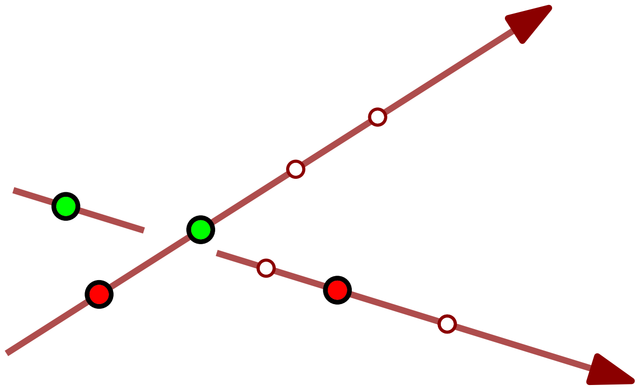
Here: which **triples** of permutations can be realized **simultaneously** in $\mathbb{R}^3$?

Simplest non-trivial constraints



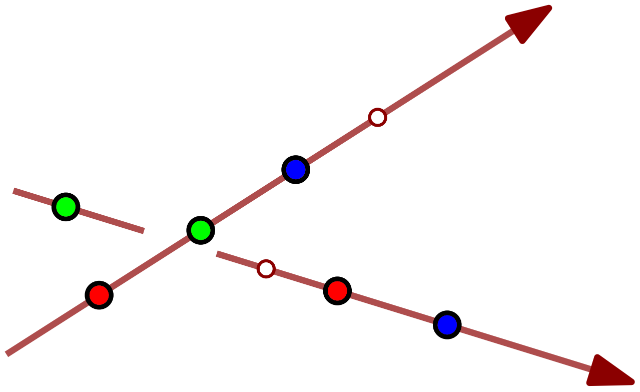
Here: which **triples** of permutations can be realized **simultaneously** in $\mathbb{R}^3$?

Simplest non-trivial constraints



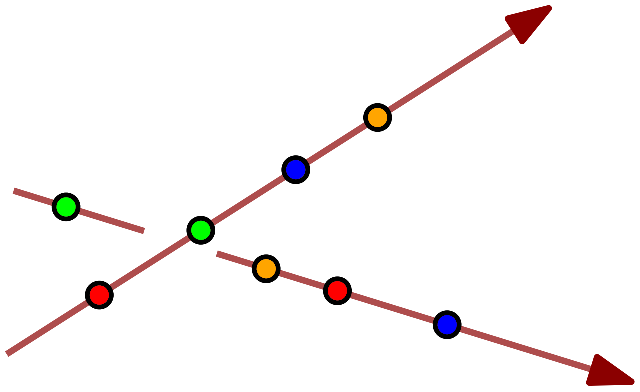
Here: which **triples** of permutations can be realized **simultaneously** in $\mathbb{R}^3$?

Simplest non-trivial constraints



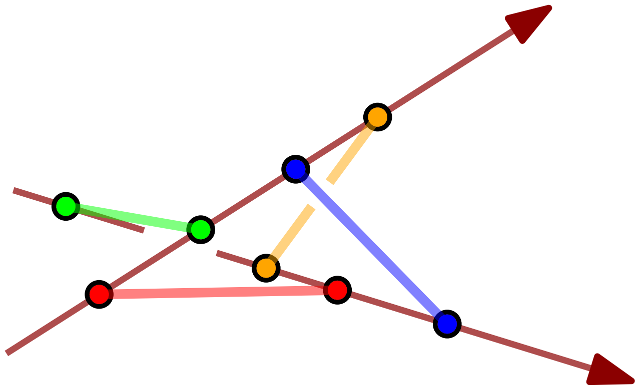
Here: which **triples** of permutations can be realized **simultaneously** in $\mathbb{R}^3$?

Simplest non-trivial constraints



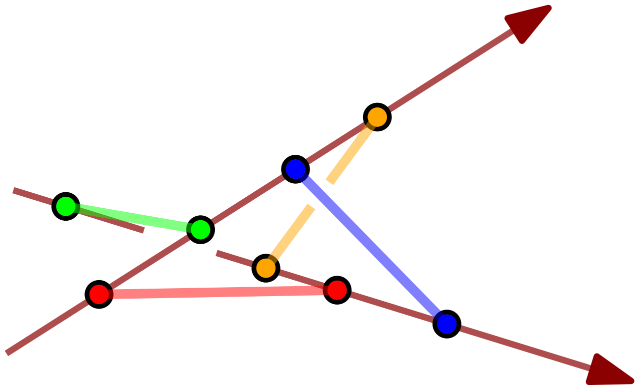
Here: which **triples** of permutations can be realized **simultaneously** in $\mathbb{R}^3$?

Simplest non-trivial constraints



Here: which **triples** of permutations can be realized **simultaneously** in $\mathbb{R}^3$?

Simplest non-trivial constraints

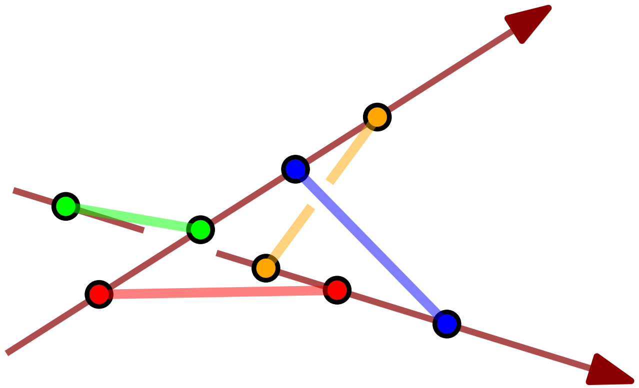


Here: which **triples** of permutations can be realized **simultaneously** in $\mathbb{R}^3$?

Simplest non-trivial constraints

We design an algorithm deciding this.

Implementation ($\sim$ 600 lines of python)



Here: which **triples** of permutations can be realized **simultaneously** in $\mathbb{R}^3$?

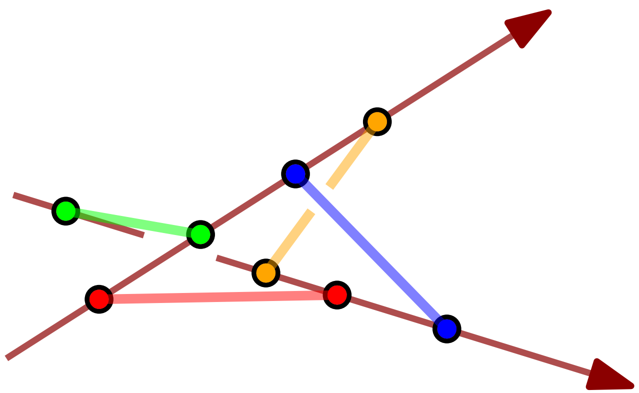
Simplest non-trivial constraints

We design an algorithm deciding this.

Implementation ($\sim$ 600 lines of python)

Experimental results:

Theorem. Every triple of permutations on $n \leq 5$ elements is geometrically realizable in $\mathbb{R}^3$.



Here: which **triples** of permutations can be realized **simultaneously** in $\mathbb{R}^3$?

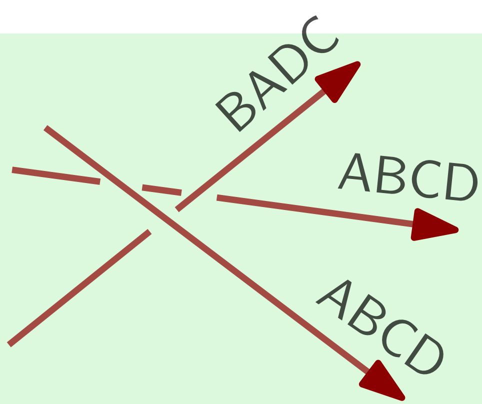
Simplest non-trivial constraints

We design an algorithm deciding this.

Implementation ($\sim$ 600 lines of python)

Experimental results:

Theorem. Every triple of permutations on $n \leq 5$ elements is geometrically realizable in $\mathbb{R}^3$.

Theorem. 

is impossible.

The computational problem

GEOMETRIC_REALISABILITY_3D

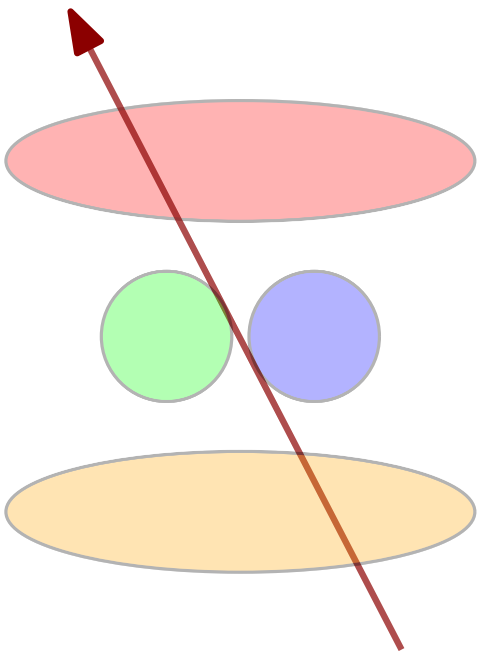
Input:	Three orders on $\{1, 2, \ldots, n\}$
Output:	Whether there exists n disjoint compact convex sets in $\mathbb{R}^3$ and three lines intersecting them in these orders.

From geometry to algebra

Geometric normalization

Start with 3 lines and n disjoint compact convex sets realizing 3 given orders.

We can assume that the lines are **pairwise skew**.

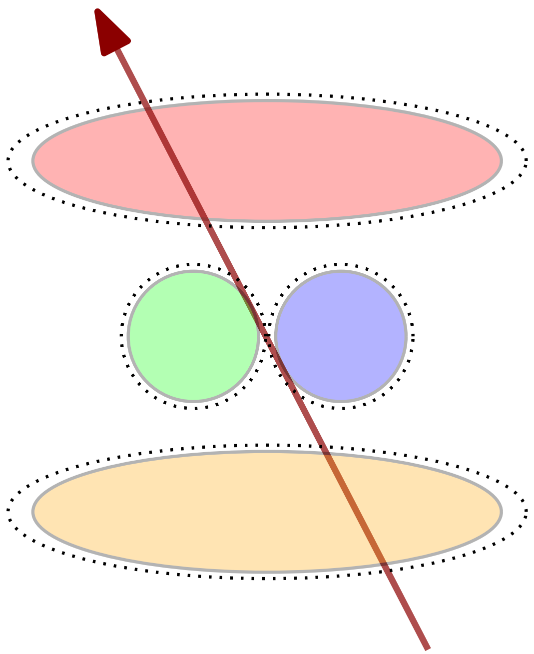


Geometric normalization

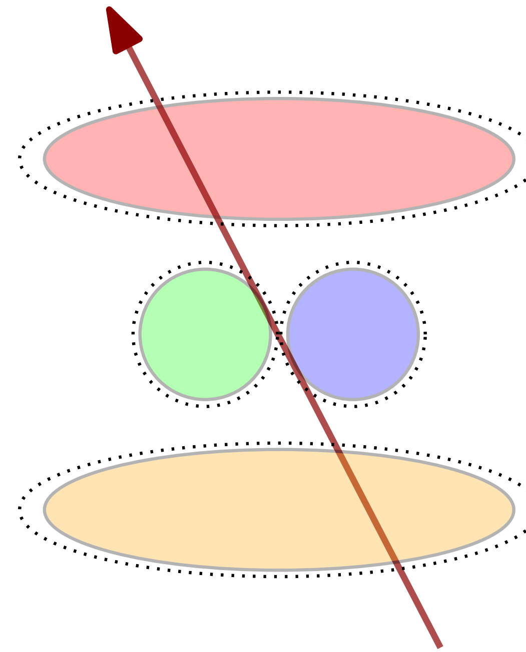
Start with 3 lines and n disjoint compact convex sets realizing 3 given orders.

We can assume that the lines are **pairwise skew**.

Compactness
$\Rightarrow$ we can thicken the sets
$\Rightarrow$ we can perturb the lines



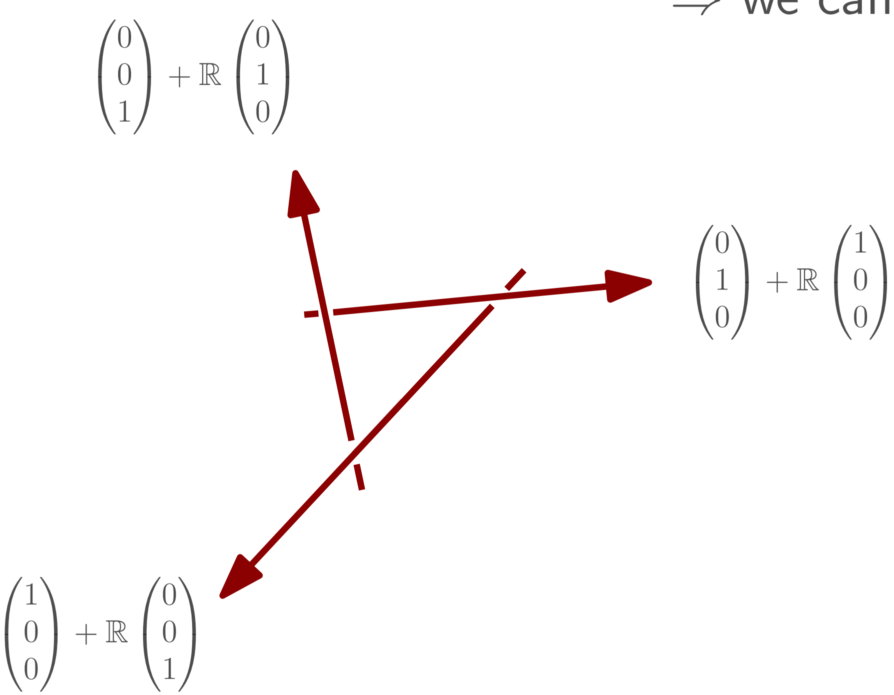
Geometric normalization



Start with 3 lines and n disjoint compact convex sets realizing 3 given orders.

We can assume that the lines are **pairwise skew**.

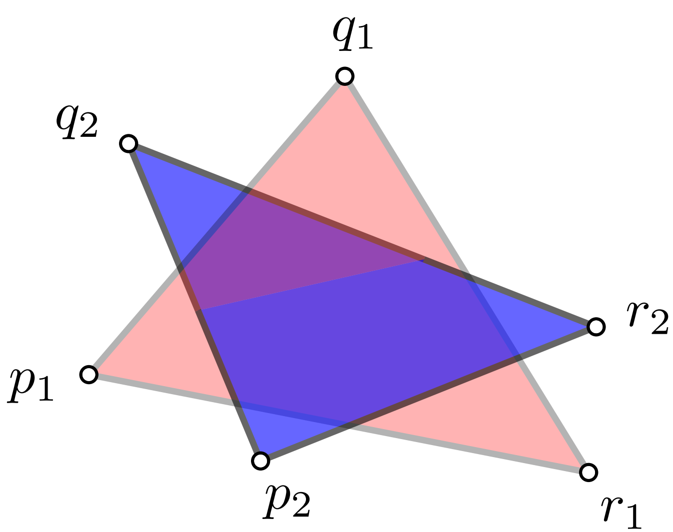
Compactness
$\Rightarrow$ we can thicken the sets
$\Rightarrow$ we can perturb the lines



We can crop the sets to **triangles** with vertices **on the lines**.

We can **choose** the lines (up to reversing some permutations).

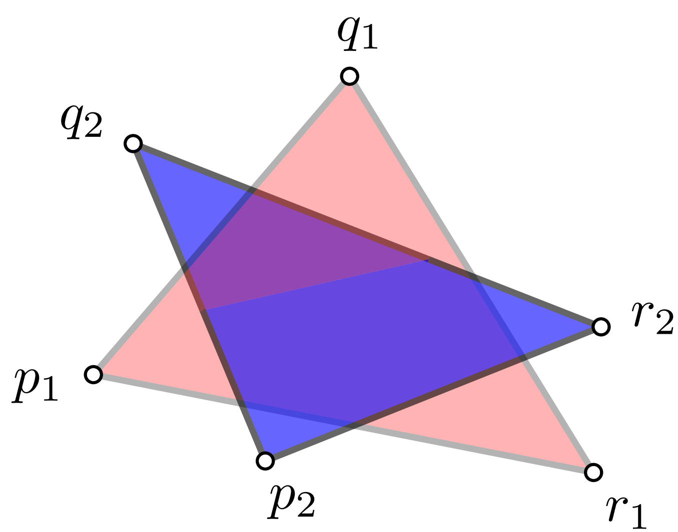
Guigue-Devillers algorithm



$$[p, q, r, s] \stackrel{\text{def}}{=} \text{sign} \begin{vmatrix} x_p & x_q & x_r & x_s \\ y_p & y_q & y_r & y_s \\ z_p & z_q & z_r & z_s \\ 1 & 1 & 1 & 1 \end{vmatrix}$$

[Journal of graphics tools, 2003]

Guigue-Devillers algorithm



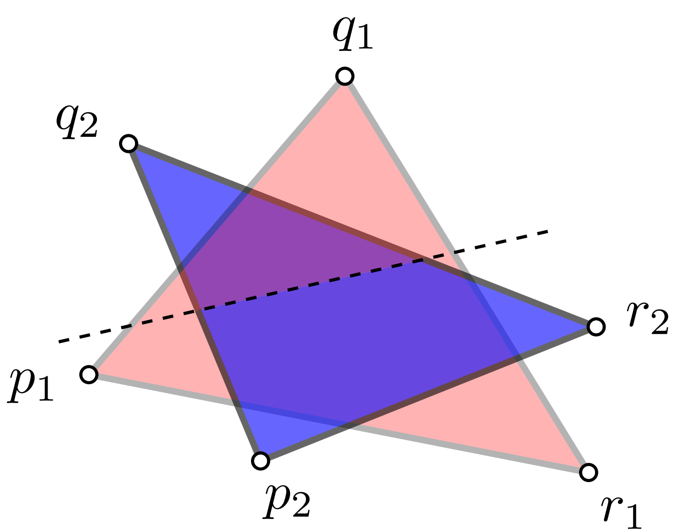
(1) does the plane of a triangle separate the vertices of the other?

$[p_1, q_1, r_1, p_2]$	$[p_2, q_2, r_2, p_1]$
$[p_1, q_1, r_1, q_2]$	$[p_2, q_2, r_2, q_1]$
$[p_1, q_1, r_1, r_2]$	$[p_2, q_2, r_2, r_1]$

$$[p, q, r, s] \stackrel{\text{def}}{=} \operatorname{sign} \begin{vmatrix} x_p & x_q & x_r & x_s \\ y_p & y_q & y_r & y_s \\ z_p & z_q & z_r & z_s \\ 1 & 1 & 1 & 1 \end{vmatrix}$$

[Journal of graphics tools, 2003]

Guigue-Devillers algorithm



(1) does the plane of a triangle separate the vertices of the other?

$[p_1, q_1, r_1, p_2]$ $[p_2, q_2, r_2, p_1]$
$[p_1, q_1, r_1, q_2]$ $[p_2, q_2, r_2, q_1]$
$[p_1, q_1, r_1, r_2]$ $[p_2, q_2, r_2, r_1]$

(2) if not, each triangle meets the intersection of the two planes.
Do the two segments intersect?

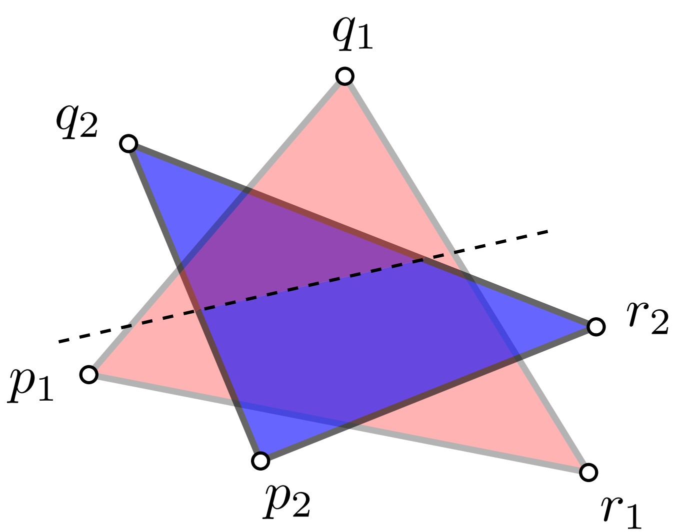
$$[p, q, r, s] \stackrel{\text{def}}{=} \operatorname{sign} \begin{vmatrix} x_p & x_q & x_r & x_s \\ y_p & y_q & y_r & y_s \\ z_p & z_q & z_r & z_s \\ 1 & 1 & 1 & 1 \end{vmatrix}$$

rename p, q, r into a, b, c so that
a is separated from $\{b, c\}$

$[a_1, b_1, a_2, b_2] = +1$ or $[a_1, c_1, c_2, a_2] = +1$

[Journal of graphics tools, 2003]

Guigue-Devillers algorithm



(1) does the plane of a triangle separate the vertices of the other?

$[p_1, q_1, r_1, p_2]$ $[p_2, q_2, r_2, p_1]$
$[p_1, q_1, r_1, q_2]$ $[p_2, q_2, r_2, q_1]$
$[p_1, q_1, r_1, r_2]$ $[p_2, q_2, r_2, r_1]$

(2) if not, each triangle meets the intersection of the two planes. Do the two segments intersect?

$$[p, q, r, s] \stackrel{\text{def}}{=} \text{sign} \begin{vmatrix} x_p & x_q & x_r & x_s \\ y_p & y_q & y_r & y_s \\ z_p & z_q & z_r & z_s \\ 1 & 1 & 1 & 1 \end{vmatrix}$$

rename p, q, r into a, b, c so that a is separated from $\{b, c\}$

$[a_1, b_1, a_2, b_2] = +1$ or $[a_1, c_1, c_2, a_2] = +1$

[Journal of graphics tools, 2003]

Semi-algebraic formulation

Parameterize **candidate realizations** by $\mathbb{R}^{3n}$

ith triangle = convex hull of $\begin{pmatrix} x_i \\ 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ y_i \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \\ z_i \end{pmatrix}$.

G-D algorithm $\leftrightarrow$ polynomial decision tree

Orientation predicates are polynomials in the parameters.

The realizations form a semi-algebraic set S defined by $\Theta\left(n^2\right)$ inequations.

Semi-algebraic formulation

Parameterize **candidate realizations** by $\mathbb{R}^{3n}$

ith triangle = convex hull of $\begin{pmatrix} x_i \\ 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ y_i \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \\ z_i \end{pmatrix}$.

G-D algorithm $\leftrightarrow$ polynomial decision tree

Orientation predicates are polynomials in the parameters.

The realizations form a semi-algebraic set S defined by $\Theta\left(n^2\right)$ inequations.

Testing emptiness of a semi-algebraic set

CAD, critical points method, ...

$(\text{number of polynomials} \times \text{maximum degree})^{O(\text{number of variables})}$

From algebra to combinatorics

Inspiration: Veronese lifting

A trick to **linearize** problems on polynomials of degree $\leq k$ in $\mathbb{R}^d$

$$(x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$$

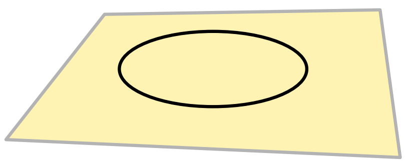
$$\mapsto \quad \begin{array}{l}(x_1, x_2, \ldots, x_d, x_1^2, x_1 x_2, \\ x_1 x_3, \ldots, x_d^2, x_1^3, \ldots, x_d^k)\end{array} \quad \in \mathbb{R}^{\binom{d+k}{d}}$$

Inspiration: Veronese lifting

A trick to **linearize** problems on polynomials of degree $\leq k$ in $\mathbb{R}^d$

$$(x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$$

$$\mapsto \quad \begin{array}{l}(x_1, x_2, \ldots, x_d, x_1^2, x_1x_2, \\ x_1x_3, \ldots, x_d^2, x_1^3, \ldots, x_d^k)\end{array} \quad \in \mathbb{R}^{\binom{d+k}{d}}$$



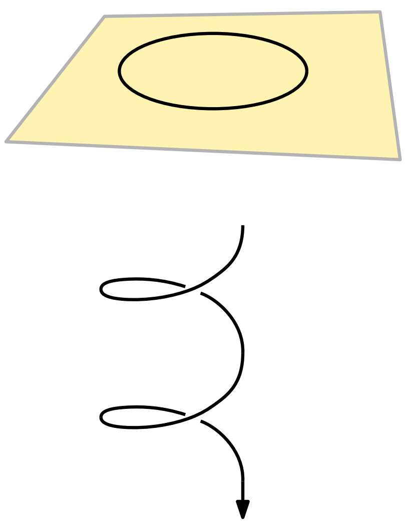
Theorem. Any d finite measures in $\mathbb{R}^d$ can be simultaneously bisected by a hyperplane.

Inspiration: Veronese lifting

A trick to **linearize** problems on polynomials of degree $\leq k$ in $\mathbb{R}^d$

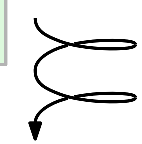
$$(x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$$

$$\mapsto \begin{array}{l} (x_1, x_2, \ldots, x_d, x_1^2, x_1x_2, \\ x_1x_3, \ldots, x_d^2, x_1^3, \ldots, x_d^k) \end{array} \in \mathbb{R}^{\binom{d+k}{d}}$$



$\mathcal{M}_2 \subset \mathbb{R}^6$

Theorem. Any d finite measures in $\mathbb{R}^d$ can be simultaneously bisected by a hyperplane.



Theorem. Any $\binom{d+k}{d}$ finite measures in $\mathbb{R}^d$ can be simultaneously bisected by the zero set of a polynomial of degree k.

Factorizing the △△ predicates

$$X_i = \begin{pmatrix} x_i \\ 1 \\ 0 \end{pmatrix}, Y_i = \begin{pmatrix} 0 \\ y_i \\ 1 \end{pmatrix} \text{ and } Z_i = \begin{pmatrix} 1 \\ 0 \\ z_i \end{pmatrix}$$

$$[p, q, r, s] = \operatorname{sign} \begin{vmatrix} x_p & x_q & x_r & x_s \\ y_p & y_q & y_r & y_s \\ z_p & z_q & z_r & z_s \\ 1 & 1 & 1 & 1 \end{vmatrix}$$

Orientation
$[X_a, X_b, Y_c, Y_d]$
$[X_a, X_b, Z_c, Z_d]$
$[Y_a, Y_b, Z_c, Z_d]$
$[X_a, X_b, Y_c, Z_d]$
$[X_a, Y_b, Y_c, Z_d]$
$[X_a, Y_b, Z_c, Z_d]$

Factorizing the △△ predicates

$X_i = \begin{pmatrix} x_i \\ 1 \\ 0 \end{pmatrix}$, $Y_i = \begin{pmatrix} 0 \\ y_i \\ 1 \end{pmatrix}$ and $Z_i = \begin{pmatrix} 1 \\ 0 \\ z_i \end{pmatrix}$

$$[p, q, r, s] = \operatorname{sign} \begin{vmatrix} x_p & x_q & x_r & x_s \\ y_p & y_q & y_r & y_s \\ z_p & z_q & z_r & z_s \\ 1 & 1 & 1 & 1 \end{vmatrix}$$

Orientation	Determinant
$[X_a, X_b, Y_c, Y_d]$	$(x_a - x_b)(y_c - y_d)$
$[X_a, X_b, Z_c, Z_d]$	$(x_a - x_b)(z_c - z_d)$
$[Y_a, Y_b, Z_c, Z_d]$	$(y_a - y_b)(z_c - z_d)$
$[X_a, X_b, Y_c, Z_d]$	$(x_a - x_b)(y_c z_d - z_d + 1)$
$[X_a, Y_b, Y_c, Z_d]$	$(y_b - y_c)(x_a - x_a z_d - 1)$
$[X_a, Y_b, Z_c, Z_d]$	$(z_c - z_d)(x_a y_b + 1 - y_b)$

Factorizing the △△ predicates

$X_i = \begin{pmatrix} x_i \\ 1 \\ 0 \end{pmatrix}$, $Y_i = \begin{pmatrix} 0 \\ y_i \\ 1 \end{pmatrix}$ and $Z_i = \begin{pmatrix} 1 \\ 0 \\ z_i \end{pmatrix}$

$$[p,q,r,s] = \operatorname{sign} \begin{vmatrix} x_p & x_q & x_r & x_s \\ y_p & y_q & y_r & y_s \\ z_p & z_q & z_r & z_s \\ 1 & 1 & 1 & 1 \end{vmatrix}$$

Orientation	Determinant	Decomposition
$[X_a, X_b, Y_c, Y_d]$	$(x_a - x_b)(y_c - y_d)$	$(x_a - x_b)(y_c - y_d)$
$[X_a, X_b, Z_c, Z_d]$	$(x_a - x_b)(z_c - z_d)$	$(x_a - x_b)(z_c - z_d)$
$[Y_a, Y_b, Z_c, Z_d]$	$(y_a - y_b)(z_c - z_d)$	$(y_a - y_b)(z_c - z_d)$
$[X_a, X_b, Y_c, Z_d]$	$(x_a - x_b)(y_c z_d - z_d + 1)$	$(x_a - x_b)(y_c - 1)\left(z_d - \frac{1}{1-y_c}\right)$
$[X_a, Y_b, Y_c, Z_d]$	$(y_b - y_c)(x_a - x_a z_d - 1)$	$-(y_b - y_c)(z_d - 1)\left(x_a - \frac{1}{1-z_d}\right)$
$[X_a, Y_b, Z_c, Z_d]$	$(z_c - z_d)(x_a y_b + 1 - y_b)$	$(z_c - z_d)(x_a - 1)\left(y_b - \frac{1}{1-x_a}\right)$

Factorizing the △△ predicates

$X_i = \begin{pmatrix} x_i \\ 1 \\ 0 \end{pmatrix}$, $Y_i = \begin{pmatrix} 0 \\ y_i \\ 1 \end{pmatrix}$ and $Z_i = \begin{pmatrix} 1 \\ 0 \\ z_i \end{pmatrix}$

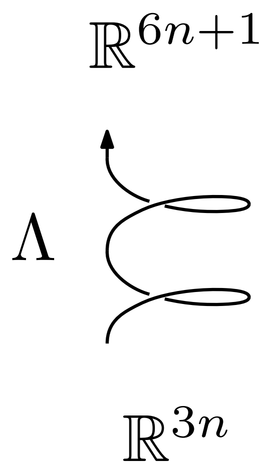
$$[p,q,r,s] = \operatorname{sign} \begin{vmatrix} x_p & x_q & x_r & x_s \\ y_p & y_q & y_r & y_s \\ z_p & z_q & z_r & z_s \\ 1 & 1 & 1 & 1 \end{vmatrix}$$

Orientation	Determinant	Decomposition
$[X_a, X_b, Y_c, Y_d]$	$(x_a - x_b)(y_c - y_d)$	$(x_a - x_b)(y_c - y_d)$
$[X_a, X_b, Z_c, Z_d]$	$(x_a - x_b)(z_c - z_d)$	$(x_a - x_b)(z_c - z_d)$
$[Y_a, Y_b, Z_c, Z_d]$	$(y_a - y_b)(z_c - z_d)$	$(y_a - y_b)(z_c - z_d)$
$[X_a, X_b, Y_c, Z_d]$	$(x_a - x_b)(y_c z_d - z_d + 1)$	$(x_a - x_b)(y_c - 1)\left(z_d - \frac{1}{1-y_c}\right)$
$[X_a, Y_b, Y_c, Z_d]$	$(y_b - y_c)(x_a - x_a z_d - 1)$	$-(y_b - y_c)(z_d - 1)\left(x_a - \frac{1}{1-z_d}\right)$
$[X_a, Y_b, Z_c, Z_d]$	$(z_c - z_d)(x_a y_b + 1 - y_b)$	$(z_c - z_d)(x_a - 1)\left(y_b - \frac{1}{1-x_a}\right)$

$$f(t) \stackrel{\text{def}}{=} \frac{1}{1-t}$$

The order on $\{x_1, f(x_1), x_2, f(x_2), \ldots, z_n, f(z_n), \mathbf{1}\}$ determines all orientations.

Combinatorial lifting



$c = (x_1, x_2, \ldots, z_n)$ a configuration

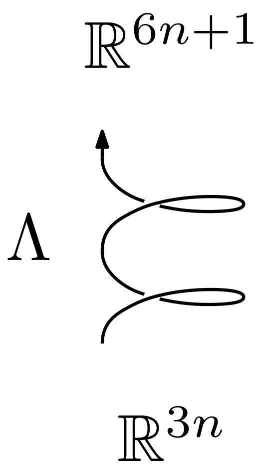
$\Lambda(c) = (x_1, f(x_1), x_2, f(x_2), \ldots, z_n, f(z_n), 1)$

$x_i \leftrightarrow t_{2i-1},\ y_i \leftrightarrow t_{2n+2i-1},\ \ldots$

$$[X_a, X_b, Y_c, Z_d] = (x_a - x_b)(y_c - 1)\,(z_d - f(y_c))$$

$$\leftrightarrow (t_{2a-1} - t_{2b-1})(t_{2n+2c-1} - t_{6n+1})(t_{4n+2d-1} - t_{2n+2c})$$

Combinatorial lifting



$c = (x_1, x_2, \ldots, z_n)$ a configuration

$\Lambda(c) = (x_1, f(x_1), x_2, f(x_2), \ldots, z_n, f(z_n), 1)$

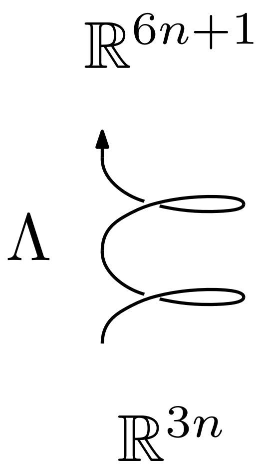
$x_i \leftrightarrow t_{2i-1},\ y_i \leftrightarrow t_{2n+2i-1},\ \ldots$

$$[X_a, X_b, Y_c, Z_d] = (x_a - x_b)(y_c - 1)\,(z_d - f(y_c))$$

$$\leftrightarrow (t_{2a-1} - t_{2b-1})(t_{2n+2c-1} - t_{6n+1})(t_{4n+2d-1} - t_{2n+2c})$$

Find an order on $(t_1, t_2, \ldots, t_{6n+1})$ that **satisfies** the symbolic lifting of the boolean formula defining S and is **realizable** by some $\Lambda(c)$.

Combinatorial lifting



$c = (x_1, x_2, \ldots, z_n)$ a configuration

$\Lambda(c) = (x_1, f(x_1), x_2, f(x_2), \ldots, z_n, f(z_n), 1)$

$x_i \leftrightarrow t_{2i-1},\ y_i \leftrightarrow t_{2n+2i-1},\ \ldots$

$$[X_a, X_b, Y_c, Z_d] = (x_a - x_b)(y_c - 1)\,(z_d - f(y_c))$$
$$\leftrightarrow (t_{2a-1} - t_{2b-1})(t_{2n+2c-1} - t_{6n+1})(t_{4n+2d-1} - t_{2n+2c})$$

Find an order on $(t_1, t_2, \ldots, t_{6n+1})$ that **satisfies** the symbolic lifting of the boolean formula defining S and is **realizable** by some $\Lambda(c)$.

The **realizability problem** becomes easy if we lift to $(x_1, f(x_1), f^{(2)}(x_1), x_2, \ldots, f^{(2)}(z_n), 0, 1)$

Combinatorial lifting

Remember: $f(t) \stackrel{\text{def}}{=} \frac{1}{1-t}$

f has nice properties:

1. $f^{(3)} = f \circ f \circ f = \text{id}$
2. f permutes circularly $(-\infty, 0)$, $(0, 1)$ and $(1, +\infty)$
3. f is increasing on each of these intervals

Combinatorial lifting

Remember: $f(t) \stackrel{\text{def}}{=} \frac{1}{1-t}$

f has nice properties:

1. $f^{(3)} = f \circ f \circ f = \text{id}$
2. f permutes circularly $(-\infty, 0)$, $(0, 1)$ and $(1, +\infty)$
3. f is increasing on each of these intervals

Theorem. An order in S_{3n+2} can be realized by $\{x_1, f(x_1), f^{(2)}(x_1), x_2, f(x_2), f^{(2)}(x_2), \ldots, z_n, f(z_n), f^{(2)}(z_n), 0, 1\}$ iff it is compatible with properties 1-2-3.

Combinatorial lifting

Remember: $f(t) \stackrel{\text{def}}{=} \frac{1}{1-t}$

f has nice properties:

1. $f^{(3)} = f \circ f \circ f = \text{id}$
2. f permutes circularly $(-\infty, 0)$, $(0, 1)$ and $(1, +\infty)$
3. f is increasing on each of these intervals

Theorem. An order in S_{3n+2} can be realized by $\{x_1, f(x_1), f^{(2)}(x_1), x_2, f(x_2), f^{(2)}(x_2), \ldots, z_n, f(z_n), f^{(2)}(z_n), 0, 1\}$ iff it is compatible with properties 1-2-3.

"the action of f"

Solving the combinatorial problem

Start with $3n$ **geometric** variables.

$\{x_i\}$, $\{y_i\}$, $\{z_i\}$, each family ordered by an input permutation

Start with $3n$ **geometric** variables.

$\{x_i\}$, $\{y_i\}$, $\{z_i\}$, each family ordered by an input permutation

Add $6n$ **symbolic** variables.

$\bullet_{i,j}$ which represents $f^{(j)}(\bullet_i)$, for $j = 1, 2$ and $\bullet \in \{x, y, z\}$.

Start with $3n$ **geometric** variables.

$\{x_i\}$, $\{y_i\}$, $\{z_i\}$, each family ordered by an input permutation

Add $6n$ **symbolic** variables.

$\bullet_{i,j}$ which represents $f^{(j)}(\bullet_i)$, for $j = 1, 2$ and $\bullet \in \{x, y, z\}$.

Insert $0, 1$ in each ordered family $\{x_i\}$, $\{y_i\}$, $\{z_i\}$.

brute force: $\Theta\left(n^6\right)$ cases.

Start with $3n$ **geometric** variables.

$\{x_i\}$, $\{y_i\}$, $\{z_i\}$, each family ordered by an input permutation

Add $6n$ **symbolic** variables.

$\bullet_{i,j}$ which represents $f^{(j)}(\bullet_i)$, for $j = 1, 2$ and $\bullet \in \{x, y, z\}$.

Insert $0, 1$ in each ordered family $\{x_i\}$, $\{y_i\}$, $\{z_i\}$.

brute force: $\Theta\left(n^6\right)$ cases.

Order each family $\{\bullet_{i,j}\}_i$ to be compatible with the action of f.

Start with $3n$ **geometric** variables.

$\{x_i\}$, $\{y_i\}$, $\{z_i\}$, each family ordered by an input permutation

Add $6n$ **symbolic** variables.

$\bullet_{i,j}$ which represents $f^{(j)}(\bullet_i)$, for $j = 1, 2$ and $\bullet \in \{x, y, z\}$.

Insert $0, 1$ in each ordered family $\{x_i\}$, $\{y_i\}$, $\{z_i\}$.

brute force: $\Theta\left(n^6\right)$ cases.

Order each family $\{\bullet_{i,j}\}_i$ to be compatible with the action of f.

A partial order $\mathcal{P}_0$ whose linear extensions contain all solution orders

Start with $3n$ **geometric** variables.

$\{x_i\}$, $\{y_i\}$, $\{z_i\}$, each family ordered by an input permutation

Add $6n$ **symbolic** variables.

$\bullet_{i,j}$ which represents $f^{(j)}(\bullet_i)$, for $j = 1, 2$ and $\bullet \in \{x, y, z\}$.

Insert $0, 1$ in each ordered family $\{x_i\}$, $\{y_i\}$, $\{z_i\}$.

brute force: $\Theta\left(n^6\right)$ cases.

Order each family $\{\bullet_{i,j}\}_i$ to be compatible with the action of f.

A partial order $\mathcal{P}_0$ whose linear extensions contain all solution orders

We then refine $\mathcal{P}$ into $\mathcal{P}_1, \mathcal{P}_2, \ldots$ such that for each $\mathcal{P}_i$, **all or none** linear extension is a solution order.

Refine/split $\mathcal{P}_i$ one pair of triangle at a time:

1st stage:

$$\begin{pmatrix} [X_1, Y_1, Z_1, X_2] \\ \dots \\ [X_2, Y_2, Z_2, Z_1] \end{pmatrix}$$

Refine/split $\mathcal{P}_i$ one pair of triangle at a time:

1st stage:

$$\begin{pmatrix} [X_1, Y_1, Z_1, X_2] \\ \ldots \\ [X_2, Y_2, Z_2, Z_1] \end{pmatrix}$$

Does $\mathcal{P}_i$ determine all orientations?

Refine/split $\mathcal{P}_i$ one pair of triangle at a time:

Are the triangles disjoint?

yes

1st stage:

$$\begin{pmatrix} [X_1, Y_1, Z_1, X_2] \\ \dots \\ [X_2, Y_2, Z_2, Z_1] \end{pmatrix}$$

Does $\mathcal{P}_i$ determine all orientations?

Refine/split $\mathcal{P}_i$ one pair of triangle at a time:

1st stage:

$$\begin{pmatrix} [X_1, Y_1, Z_1, X_2] \\ \dots \\ [X_2, Y_2, Z_2, Z_1] \end{pmatrix}$$

Does $\mathcal{P}_i$ determine all orientations?

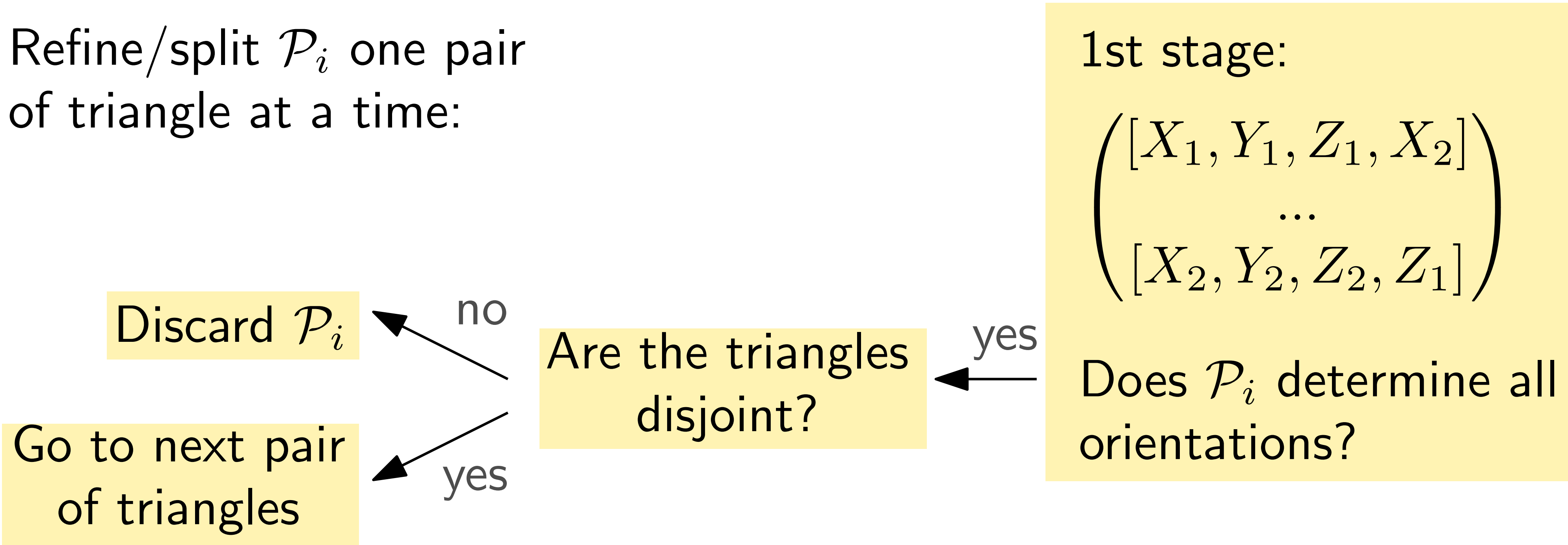
yes

Are the triangles disjoint?

no

Discard $\mathcal{P}_i$

Refine/split $\mathcal{P}_i$ one pair of triangle at a time:



Refine/split $\mathcal{P}_i$ one pair of triangle at a time:

1st stage:

$$\begin{pmatrix} [X_1, Y_1, Z_1, X_2] \\ \dots \\ [X_2, Y_2, Z_2, Z_1] \end{pmatrix}$$

Does $\mathcal{P}_i$ determine all orientations?

yes

Are the triangles disjoint?

no

Discard $\mathcal{P}_i$

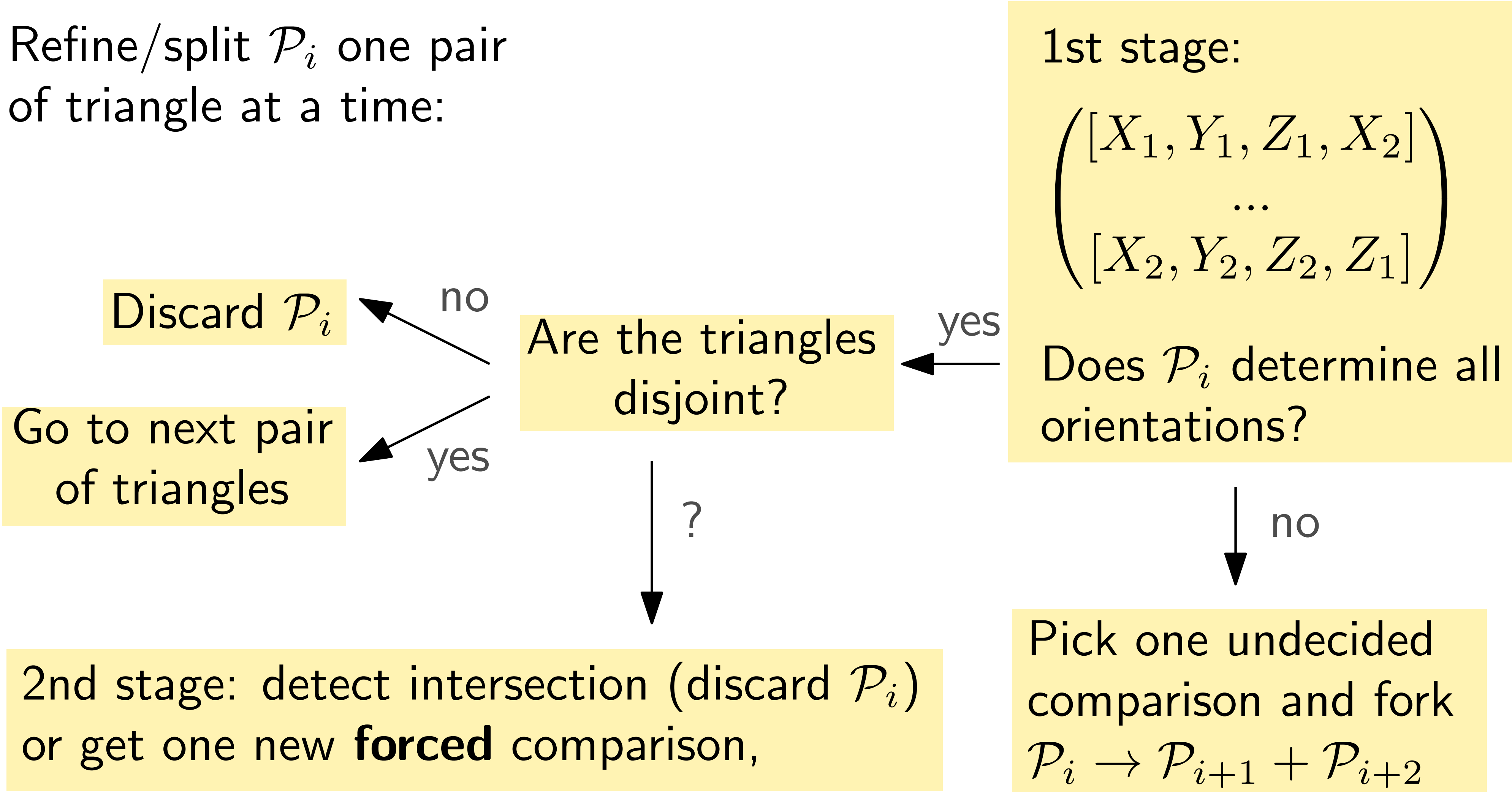
yes

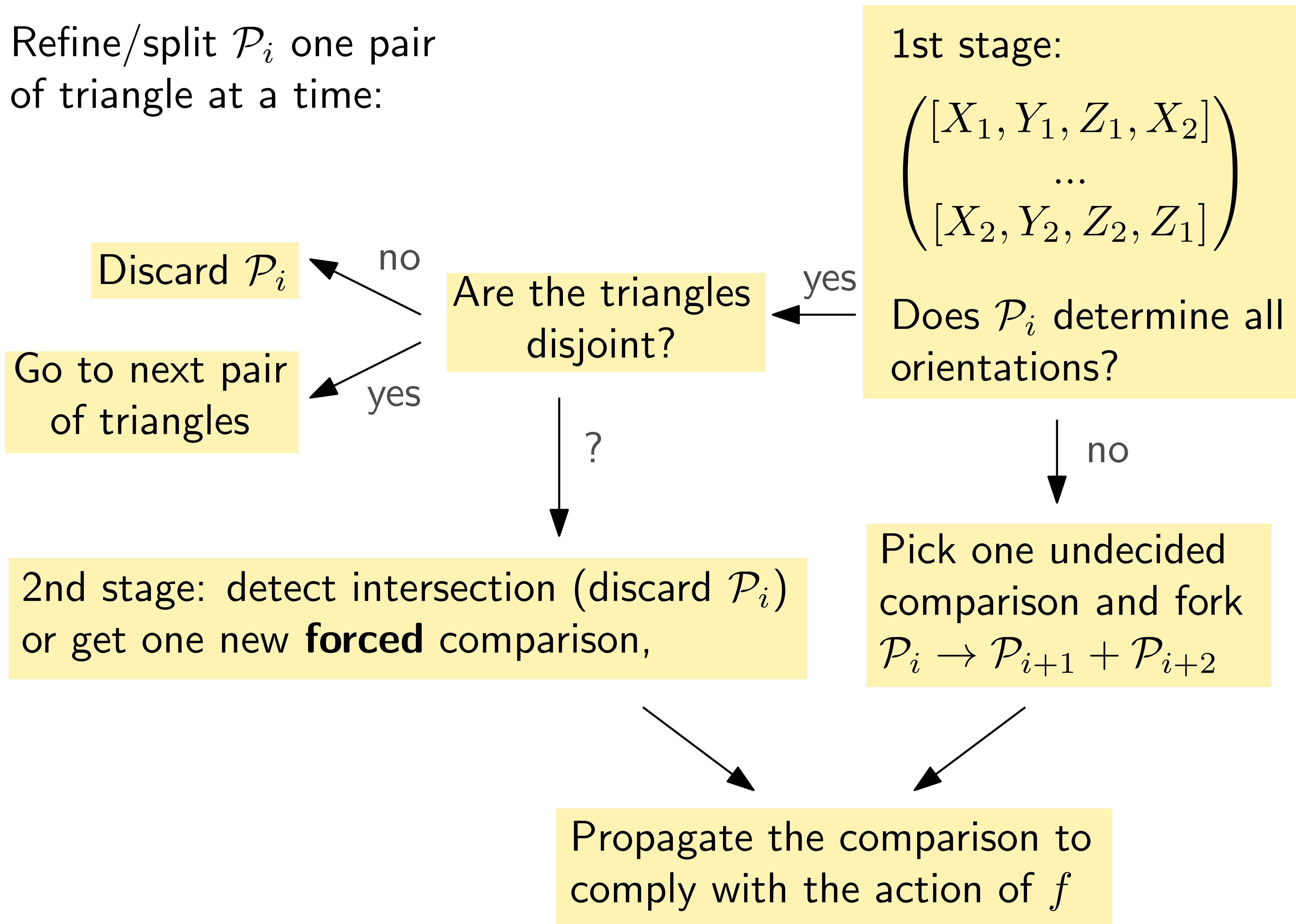
Go to next pair of triangles

?

2nd stage: detect intersection (discard $\mathcal{P}_i$) or get one new **forced** comparison,

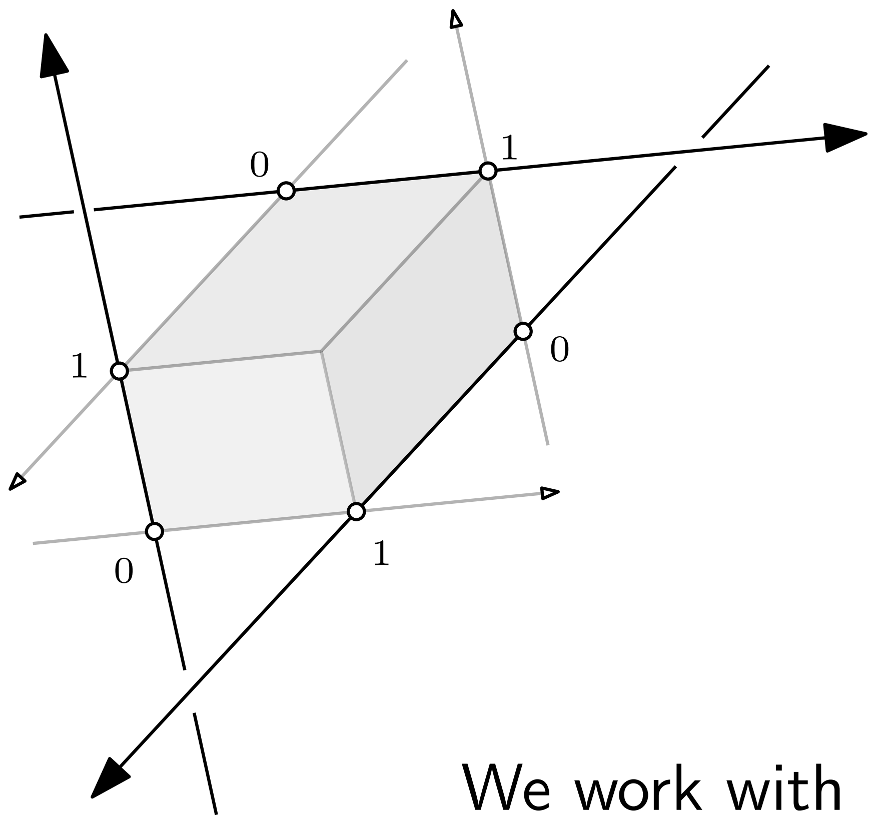
Refine/split $\mathcal{P}_i$ one pair of triangle at a time:





Refine/split $\mathcal{P}_i$ one pair of triangle at a time:
1st stage:
$$\begin{pmatrix} [X_1, Y_1, Z_1, X_2] \\ \dots \\ [X_2, Y_2, Z_2, Z_1] \end{pmatrix}$$
Does $\mathcal{P}_i$ determine all orientations?
yes
no
Are the triangles disjoint?
no
yes
?
Discard $\mathcal{P}_i$
Go to next pair of triangles
2nd stage: detect intersection (discard $\mathcal{P}_i$) or get one new **forced** comparison,
Pick one undecided comparison and fork $\mathcal{P}_i \to \mathcal{P}_{i+1} + \mathcal{P}_{i+2}$
Propagate the comparison to comply with the action of f

Using the implementation



"*Insert* $0, 1$ *in each ordered family* $\{x_i\}$, $\{y_i\}$, $\{z_i\}$"

has a geometric meaning...

We work with permutations **tagged** with 0 and 1.

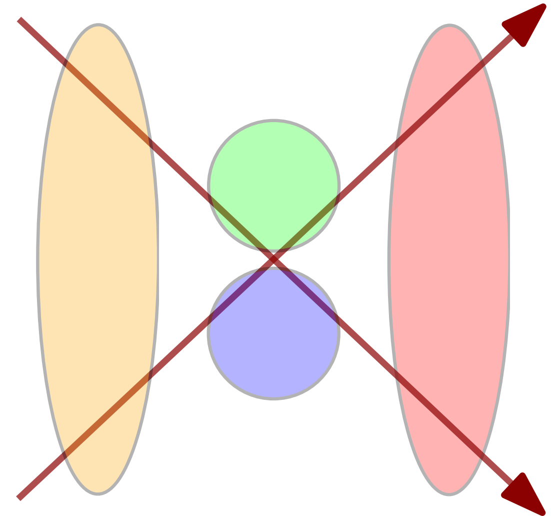
We compute the **minimally forbidden** triples of tagged permutations of size 2 to 6.

Size 2: equivalent to a lemma of [Asinowski-Katchalski 2005]

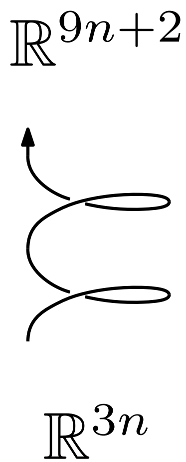
Size 5 and 6: none

To summarize

A "hard nut" in discrete geometry.

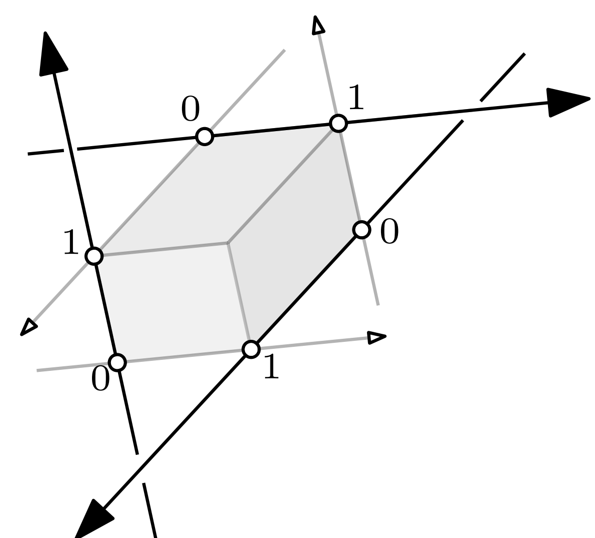


Standard reduction to a the emptiness of a semi-algebraic set.



Unexpected factorization through a function f with nice properties.

Allows to test emptiness **combinatorially**.



New geometric results

$+$ reveals some useful structure