Approximating $k$-fold filtrations using weighted Delaunay triangulations

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$k$-cover
$k$-cover

$k$-fold filtration
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There are two approaches to $k$-fold filtrations:
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- **Classical persistence**: Fix $k$ and let the radius of balls increase.

  - Algorithmic solutions using rhomboic tiling proposed by Edelsbrunner and Osang that are not trivial.
  - Can we approximate the result using union of balls?
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- Persistence in depth: Fix the radius and let $k$ decrease.
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- Classical persistence: Fix $k$ and let the radius of balls increase.
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Can we approximate the result using union of balls?
Starting easy...  \( k = 1 \)
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Classical data structures

- Čech complex.
- Alpha complex.
Classical data structures

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Restricting the growth of balls to the cells of the Voronoi diagram.
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Restricting the growth of balls to the cells of the Voronoi diagram.

Filtration on the Delaunay triangulation
A functional point of view

\[ d(x) = \min_{p \in P} ||x - p|| \]
A functional point of view

\[ d(x) = \sqrt{\min_{p \in P} ||x - p||^2 + 0} \]
A functional point of view

\[ d(x) = \sqrt{\min_{p \in P} ||x - p||^2 + w^2} \]
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A functional point of view

\[ d(x) = \sqrt{\min_{p \in P} \|x - p\|^2 + w^2} \]
A functional point of view

\[ d(x) = \sqrt{\min_{p \in P} \| x - p \|^2 + w^2} \]
Now for $k = 2$
Now for $k = 2$
Now for $k = 2$
Now for $k = 2$
Zooming on the lense
Zooming on the lense
Zooming on the lense

A

B

$k$-fold filtration
Zooming on the lense
Expressing this ball

Using previous work: distance to measure ($k$-distance).
Expressing this ball

Using previous work: distance to measure \((k\text{-distance})\).

Take the middle point \(C = A + B\) and consider the power distance:

\[
f(x) = \sqrt{||C - x||^2 + \frac{||A - B||^2}{4}}
\]
Expressing this ball

Using previous work: distance to measure ($k$-distance).

Take the middle point $C = A + B$ and consider the power distance:

$$f(x) = \min_{A,B} \sqrt{||C - x||^2 + \frac{||A - B||^2}{4}}$$
Manicheism is too simple
Going to $k = 3$
Going to $k = 3$
Going to $k = 3$
Going to $k = 3$
Considering every set of $k$-points $x_1, \ldots, x_k$ we build the barycentre $b$ and compute the weight $w^2 = \frac{1}{k} \sum \| b - x_i \|^2$. 
\textit{k}-distance and barycentres

- Considering every set of \( k \)-points \( x_1, \ldots, x_k \) we build the barycentre \( b \) and compute the weight \( w^2 = \frac{1}{k} \sum ||b - x_i||^2 \).

- Then we look at the sublevel sets of the power distance:

\[
f(x) = \min_{x_1, \ldots, x_k} \sqrt{||x - b||^2 + w^2}
\]
Barycentric over-approximation
Taking the nerve of the union, we have a filtered simplicial complex.
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We have a large number of balls but we can restrict to the cells of the $k$-order Voronoi diagram.
Taking the nerve of the union, we have a filtered simplicial complex.

We have a large number of balls but we can restrict to the cells of the $k$-order Voronoi diagram.

The result is a weighted Delaunay triangulation.
Resulting triangulation
Theorem

\[ \forall k \geq 2, \ \forall \alpha > 0, \ K_{\alpha}^{(k)} \subset A_{\alpha} \subset K_{\sqrt{k\alpha}}^{(k)} \]
Location of barycentres

$x_1$  

$x_2$  

$x_3$
Location of barycentres

$\begin{align*}
  x_1 \\
  x_2 \\
  x_3
\end{align*}$
Location of barycentres
Location of barycentres

\[ b = x_1 + x_2 + x_3 \]
Location of barycentres
Replace barycenters by the center of the minimum enclosing ball.

\[ f(x) = \min_{T = x_1, \ldots, x_k} ||x_s T||^2 + R^2 \]
Minimum enclosing balls

- Replace barycenters by the center of the minimum enclosing ball.

- Replace the weight by the radius $R$ of the minimum enclosing ball.
Minimum enclosing balls

- Replace barycenters by by the center of the minimum enclosing ball.

- Replace the weight by the radius $R$ of the minimum enclosing ball.

$$f(x) = \min_{T=x_1,...,x_k} \sqrt{||x - s_T||^2 + R_T^2}$$
Minimum enclosing balls covering
Theorem

\[ \forall k \geq 2, \, \forall \alpha > 0, \, \mathcal{K}_\alpha^{(k)} \subset \mathcal{E}_\alpha \subset \mathcal{K}_{\sqrt{2\alpha}}^{(k)} \]
The conjecture

Conjecture

Every non-empty cell in the MEB-diagram is also a non-empty cell in the k-order Voronoi diagram.
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Conjecture

Every simplex of the MEB-triangulation is also a simplex of the k-order Delaunay triangulation.
It is possible to approximate the $k$-fold filtration within a constant approximation factor using a weighted Delaunay triangulation.
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We need to use a triangulation based on minimum enclosing balls rather than barycentres.
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We need to use a triangulation based on minimum enclosing balls rather than barycentres.

This triangulation seems smaller than the $k$-order Delaunay triangulation.