Scaffolding skeletons using spherical Voronoi diagrams: feasibility, regularity and symmetry.

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Outline

1. Introduction

2. Scaffolds

3. Models & Existence

4. Algorithms
Outline

1 Introduction

2 Scaffolds

3 Models & Existence

4 Algorithms
Premises

**Skeleton**: finite set of spatial line segments that do not intersect except at endpoints.

The skeleton $S$ naturally defines a graph $G_S = (\mathcal{E}_S, \mathcal{V}_S)$ embedded in $\mathbb{R}^3$.

**Scaffold**: coarse quad mesh that *tightly follows* the structure of the skeleton. (informal definition)
Motivation: an intermediate step in many applications

- Sculpting [JLW10]
- Subdivision surface [BMW12]
- Compatible quadrangulation [YCJL09]
- Semi-regular quad meshing [ULP^15]
- Architecture [SMA05]
- Bi-quartic surfaces [KP16]
Previous work

1. Construct a pipe with a polygonal cross profile, then “stitch” the pipes at the joints [SMA05, JLW10].

2. Partition a cube at joints, then extrude quadrilateral “tubes” connecting the extremities of each edge [YCJL09, ULP+15].

3. Partition a sphere at joints into cells, then construct a tubular structure connecting the two cells of each edge [BMW12].
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Difficulties in previous methods

Usai et al. [ULP+15] & Yao et al. [YCJL09]:
“lids”, spurious quads around joints

Problem: “lid” position, extra quads

Our solution
Difficulties in previous methods

Skeleton without cycles

Skeleton with a cycle

- Cycles.
- Symmetries.
- Optimality.
Difficulties in previous methods

Skeleton without cycles

Skeleton with a cycle

Contributions
- Cycles.
- Symmetries.
- Optimality.
Why symmetry?

Scaffold (no symmetries)

Symmetric scaffold (many symmetries)

Rotation symmetry

*Similar scaffolds were used in [KP16] for the construction of bi-quartic surfaces.
Scaffolding method: the outline

**Skeleton to Quad dominant Mesh (SQM)** method [BMW12].

**Construction of a scaffold as a three-step process**

1. Partition spheres at joints, one region per incident edge.
   - spherical Voronoi diagrams.

2. Discretize regions into cells (points on the boundary).
   - subdivide the boundary of a Voronoi region into a polyline.

3. Link points on the cells relative to the same edge.
   - pair points minimizing length.

**Constraint**

The two cells of every edge must have the same number of points.
Scaffolding method: the outline

**Skeleton to Quad dominant Mesh (SQM)** method [BMW12].

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### Construction of a scaffold as a three-step process

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### Constraint

The two cells of every edge must have the same number of points.
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Scaffold cells

- \( S \sim G_S(V_S, E_S) \)
- \( v \in V_S \) \( e \in E_S \)
- \( e \) incident to \( v \)

\( S_v \) sphere centered at \( v \)

\( A_v = \{ e \cap S_v | e \rightarrow v \} \)

\( R^e_v \) region around \( e \cap S_v \)

\( C^e_v \) cell discretization of \( R^e_v \)

\( \text{Vor}(A_v) \) Voronoi diagram of \( A_v \)
Scaffold, formalization

**Scaffold:** is a pair \((P_S, \Phi_S)\), such that

1. \(P_S = \{C_v \mid v \in V_S\}\), where each \(C_v = \{C^v_e \mid e \in E_S, e \rightarrow v\}\) is a family of *cells* representing a partition of \(S_v\) according to \(\text{Vor}(A_v)\).

2. \(\Phi_S = \{\phi_e \mid e \in E_S\}\) is a family of bijections \(\phi_e\) between \(C^a_e\) and \(C^b_e\) for \(e = ab\).

\[
\text{Quad} \langle p_1, \phi_e(p_1), \phi_e(p_2), p_2 \rangle
\]

\[
C^a_e = \langle p_0, p_2, \ldots, p_n \rangle, \text{ gives quads } \langle p_i, \phi_e(p_i), \phi_e(p_i+1), p_i+1 \rangle \text{ for } i = 1, 2 \ldots n.
\]
Regularity & Symmetry

Regular scaffold: all the cells have the same number of points.

Symmetric scaffold: respects all the symmetries $T \in \mathcal{T}_S$.

Skeleton symmetry: an isometry $T : \mathbb{R}^3 \to \mathbb{R}^3$ that maps elements of $G_S(\mathcal{V}_S, \mathcal{E}_S)$ to elements of $G_S$.

A scaffold respects the skeleton symmetry $T$ if:
- Symmetric cells: $C_{T(v)}^{T(v)} = T(C_v^v)$ for all cells $C_v^v$.
- Symmetric links: $\phi_{T(e)} = T \circ \phi_e \circ T^{-1}$ for all edges $e$.

$\mathcal{T}_S$ is the group generated by some symmetries of $S$. 
Regular scaffold: all the cells have the same number of points.

Symmetric scaffold: respects all the symmetries \( T \in \mathcal{T}_S \).

Skeleton symmetry: an isometry \( T : \mathbb{R}^3 \to \mathbb{R}^3 \) that maps elements of \( G_S(V_S, E_S) \) to elements of \( G_S \).

A scaffold respects the skeleton symmetry \( T \) if:

- **Symmetric cells:** \( C_{T(v)}^{T(e)} = T(C_v^e) \) for all cells \( C_v^e \).
- **Symmetric links:** \( \phi_{T(e)} = T \circ \phi_e \circ T^{-1} \) for all edges \( e \).

\( \mathcal{T}_S \) is the group generated by some symmetries of \( S \).
Regularity vs Symmetry

Asymmetric cells

Regular asymmetric

Symmetric irregular.
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Spherical Voronoi diagram and arc subdivisions

- **Del(\(A_v\))**: Delaunay triangulation of \(A_v\), dual of Vor(\(A_v\)), equivalent to the convex hull of \(A_v\) [GM01].
- \(E_v\) edges of Del(\(A_v\)) ≡ arcs on the boundaries of Vor(\(A_v\)).
- \(x_f^v\) number of segments in the subdivision of the arc \(f \in E_v\).
- \(|C_e^v| = \sum_{f \in E_v} x_f^v\) the number of points in the cell \(C_e^v\).

Cells from Vor(\(A_v\)).
Convex hull of \(A_v\).
Del(\(A_v\)).
Compatibility constraints & Model

- \(|C^a_e| = |C^b_e|\) for each skeleton edge \(e = ab\) (Compatibility constraint).
- \(x^u_f \geq 1\) each arc on the boundary of Voronoi regions must be represented by at least one segment.
- \(\Lambda_i(x^u_f) \geq s_i\) extra constraints imposed on the minimal number of points in cells or arcs (linear forms on \(x^u_f\) with nonnegative coefficients, and constants \(s_i > 0\)).

Model

\[
\begin{align*}
\sum_{h \in E_a} x^a_h = \sum_{g \in E_b} x^b_g & \quad \forall e = ab \in E_S. \\
\end{align*}
\]

\[
\begin{align*}
& x^u_f \in \mathbb{Z}, \quad x^u_f \geq 1 & \forall f \in E_v, v \in V_S \\
& \Lambda_i(x^u_f) \geq s_i & i = 1, 2, \ldots
\end{align*}
\]
Some choices for $\Lambda_i$

- Minimal number of points on cells.

\[ |C^v_e| \geq 4 \]
\[ x^v_f \geq 1 \]

\[ |C^v_e| \geq 3 \]
\[ x^v_f \geq 1 \]

- Minimal number of segments on arcs.

\[ x^v_f \geq 1 \]
\[ |C^v_e| \geq 4 \]

\[ x^v_f \geq 2 \]
\[ |C^v_e| \geq 4 \]

degenerate case!
Regular scaffold

- $|C'_e| = q$ for all the skeleton edges $e$.
- $q$ integer variable, number of points on each cell (cross profile: $q = 4$ quadrilateral, $q = 3$ triangular...)

Model (Regular)

\[
\left\{
\begin{array}{l}
\sum_{h \in E_v} x^v_h = q & \forall v \in V_S, e \in E_S, e \rightarrow v. \\
 x^v_f, q \in \mathbb{Z}, x^v_f \geq 1 & \forall f \in E_v, v \in V_S \\
 \Lambda_i(x^v_f) \geq s_i & i = 1, 2, ...
\end{array}
\right.
\]
Symmetric model

Voronoi diagram commutes with symmetry:

\[
\begin{align*}
\mathcal{A}_v & \xrightarrow{T} \mathcal{A}_{T(v)} \\
\downarrow \text{Vor} & \quad \quad & \downarrow \text{Vor} \\
\text{Vor}(\mathcal{A}_v) & \xrightarrow{T} T(\text{Vor}(\mathcal{A}_v)) = \text{Vor}(\mathcal{A}_{T(v)})
\end{align*}
\]

In a symmetric scaffold:

\[
x_f^v = x_{T(f)}^{T(v)} \quad \forall T \in \mathcal{T}_S.
\]

This condition is also **sufficient** if the arcs are discretized into equal-length segments.
Existence of scaffolds

Theorem (Existence of regular symmetric scaffolds)

Given a skeleton $S$ admitting the set of symmetries $T_S$, there exist a solution $(\bar{x}_v^f, \bar{q})$ to

$$\begin{cases} 
\sum_{h \in E_v \overset{h \circ (S_v \cap e)}{\to} (S_v \cap e)} x_h^v = q & \forall v \in V_S, e \in E_S, e \circ v. \\
x_v^f, q \in \mathbb{Z}, \ x_v^f \geq 1 & \forall f \in E_v, v \in V_S \\
\Lambda_i(x_v^f) \geq s_i & i = 1, 2, \ldots
\end{cases}$$

Satisfying

$$x_v^f = x_{T(v)}^{T(f)} \quad \forall T \in T_S, v \in V_S, f \in E_v.$$
Existence of scaffolds (preliminary results)

Lemma (Locally uniform discretization)

For $v \in \mathcal{V}_S$, the \textit{local} system
\[
\sum_{f \in \mathcal{E}_v} x^v_f = \lambda_v \quad \forall e \rightarrow v, e \in \mathcal{E}_S
\]
has a solution $(\tilde{x}^v_f, \tilde{\lambda}_v)$ with positive integer entries.

Proof.

- $\text{Del}(\mathcal{A}_v)$ is equivalent to the convex hull of $\mathcal{A}_v$, which is an inscribed polyhedron.
- \textbf{Positive real} solution with $\lambda_v = 1$ is guaranteed by a numerical characterization of graphs of inscribable type due to Rivin [Riv96].
- A homogeneous linear system with integer coefficients has a positive integer solution whenever it has a positive real solution.
Existence of scaffolds (proof)

Proof of theorem (Existence of scaffolds).

- For each node $v \in \mathcal{V}_S$ take the local solution $(\tilde{x}_f^v, \tilde{\lambda}_v)$ guaranteed by the locally uniform discretization lemma.
- Multiply each local solution by a (different) positive integer such that all the cells have the same number of points and the $\Lambda_i$ constraints are satisfied.
  \[
  \hat{x}_f^v = s \frac{\hat{\lambda}}{\tilde{\lambda}_v} \tilde{x}_f^v, \text{ where } \hat{\lambda} = \prod_{u \in \mathcal{V}_S} \tilde{\lambda}_u \text{ and } s = \max_i s_i.
  \]
  \[
  |\hat{C}_e^v| = s\hat{\lambda} \text{ holds, the factors } s\hat{\lambda}/\tilde{\lambda}_v \text{ guarantee the } \Lambda_i \text{ constraints.}
  \]
- Once we have a regular solution we can symmetrize it by summing over the orbit of symmetries.
  \[
  \bar{x}_f^v = \sum_{T \in \mathcal{T}_S} \hat{x}^{T(v)}_{T(f)}, \text{ then}
  \]
  \[
  \bar{x}^{T(v)}_{T(f)} = \sum_{R \in \mathcal{T}_S} \hat{x}^{RT(v)}_{RT(f)} = \sum_{R \in \mathcal{T}_S} \hat{x}^{R(v)}_{R(f)} = \bar{x}_f^v.
  \]
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  \]
Existence of scaffolds (proof)

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  \bar{x}_T^v \bar{x}_T^f = \sum_{R \in \mathcal{T}_S} \hat{x}_{RT}^v \hat{x}_{RT}^f = \sum_{R \in \mathcal{T}_S} \hat{x}_R^v \hat{x}_R^f = \bar{x}_f^v.
  \]
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  \]
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Computing a scaffold using Integer Linear Programming

- Optimal solution minimizing the number of quads can be found using Integer Linear Programming with the objective function:

\[
Q = \sum_{v \in (V_S - L_S)} \sum_{f \in E_v} 2x^v_f + \sum_{v \in L_S} \sum_{f \in E_v} x^v_f,
\]

- $L_S$ nodes of $G_S$ with only one incident edge.
- Number of quads in the scaffold given by

\[
\frac{1}{2} (Q + \Xi_S),
\]

- $\Xi_S$ a constant that only depends on the skeleton $S$.
- Existence of Scaffolds guarantees a solution.
General algorithm for constructing a scaffold

**Input:** The set of nodes $\mathcal{V}_S$ and edges $\mathcal{E}_S$ of the skeleton.

**Output:** The quads that represent a scaffold.

1. For each node $v \in \mathcal{V}_S$ define:
   - $A_v$ the intersection of $S_v$ with edges incident to $v$.
   - $H_v$ convex hull of $A_v$.
   - $E_v$ edges of $H_v$.

2. Define and solve the linear program on $x^v_f$ that gives compatible cells.

3. Compute points on each cell. *(Subdivide arcs into equal-length cords)*

4. Define bijections of linked cells. *(Minimizing total length)*

5. Output the quads of the scaffold:
   - For each edge $e = ab$, let $C_e^a = \langle p_0, p_2, \ldots, p_n \rangle$, for $i = 1, 2 \ldots n$:
     - Output quad $\langle p_i, \phi_e(p_i), \phi_e(p_{i+1}), p_{i+1} \rangle$
Algorithm to compute subdivisions

**Input:** Nodes $\mathcal{V}_S$ and edges $\mathcal{E}_S$, along with $E_v$ for each $v \in \mathcal{V}_S$.

**Output:** $x^v_f$ representing the subdivisions for each arc.

1. Initialize the linear program $IP$.
2. For each node $v \in \mathcal{V}_S$, and edge $f \in E_v$:
   - Add integer variable $x^v_f$ to $IP$ with restriction $x^v_f \geq 1$.
   - If the arc associated to $x^v_f$ has length $\geq \frac{5\pi}{6}$: Add restriction $x^v_f \geq 2$.
3. For each cell $C^v_e$:
   - Add: $\sum_{f \in E_v \atop f \rightarrow (e \cap \mathcal{S}_v)} x^v_f \geq 4$. (At least quadrangular cross profile)
4. For each edge $e \in \mathcal{E}_S$:
   - Add: $\sum_{g \in E_a \atop g \rightarrow (e \cap \mathcal{S}_v)} x^a_g = \sum_{h \in E_b \atop h \rightarrow (e \cap \mathcal{S}_v)} x^b_h$. (For regularity two eq. with RHS $q$)
   - For symmetry there are other extra restrictions.
5. Solve $IP$ minimizing: $\sum_{v \in (\mathcal{V}_S - L_S)} \sum_{f \in E_v} 2x^v_f + \sum_{v \in L_S} \sum_{f \in E_v} x^v_f$. 

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Thank you

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<table>
<thead>
<tr>
<th>Reference</th>
<th>Title and Authors</th>
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<tbody>
<tr>
<td>[ULP+15]</td>
<td>Francesco Usai, Marco Livesu, Enrico Puppo, Marco Tarini, and Riccardo Scateni,</td>
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Implementation

Libraries

**Skelton:** C++, (LGPL)
https://gitlab.inria.fr/afuentes/skelton

**PySkelton:** Python, (AGPL)
https://gitlab.inria.fr/afuentes/pyskelton

General workflow:

- Create or load a skeleton graph.
- Define properties of the scaffold (min cell quads, max arc angle, symmetries).
- Compute scaffold.
- Show/Save output.
Material design

- Highly symmetric shape.
- Surface mesh.
- Volumetric mesh.

(From [PZM+15, PRZ17])
Material design
Rotation symmetry

Other skeletons with cycles

Random skeleton
### Timings

<table>
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<tr>
<th>Component</th>
<th>Time</th>
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<tbody>
<tr>
<td>Total</td>
<td>1398 ms</td>
</tr>
<tr>
<td>LP Solver</td>
<td>37 ms</td>
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<tr>
<td>Convex Hulls</td>
<td>209 ms</td>
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<tr>
<td>Other</td>
<td>1152 ms</td>
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<tr>
<td>LP Solver (reg.)</td>
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### Scaffold

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<tr>
<td>Edges</td>
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<tr>
<td>Quads</td>
<td>3327</td>
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<tr>
<td>Quads (reg.)</td>
<td>13310</td>
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<tr>
<td>Cross prof. (reg.)</td>
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### Linear Programs

<table>
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